No Correlations Involved: Decision Making Under Uncertainty in the Conservative Information Space
Supplementary Material

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This document provides supplementary material to the paper [1]. Therefore, it should not be considered a self-contained document, but instead regarded as an appendix of [1], and cited as:

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Throughout this report, standard notations are used to refer to equations from [1] (e.g. Eq. (13)), while equations introduced herein are represented by the corresponding Appendix letter, e.g. Eqs. (A1) and (B21).

This document is organized as follows: Appendices A and B provide proofs for Lemmas 1 and 2; Appendix C proves Conjecture 2 for two specific basic cases ($n = 2$ and $n = 3$); Appendix D provides additional numerical results, considering a high-dimensional decision making problem ($X \in \mathbb{R}^{1600}$).

Appendix A: Proof of Lemma 1

Before presenting the proof of Lemma 1, we recall Givens rotations and introduce notations that will be used in the proof.

Givens rotations is one possible approach to update an existing square root information matrix $R \in \mathbb{R}^{n\times n}$ with the Jacobian A, i.e. to calculate the *a posteriori* information matrix R^+:

$$\begin{bmatrix} R \\ A \end{bmatrix} \rightarrow R^+ \in \mathbb{R}^{n\times n}. \tag{A1}$$

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One proceeds by applying Givens rotations to nullify all entries of A while the entries of R are appropriately updated. To see that, we consider the first Givens rotation:

$$\begin{bmatrix} c & & -s \\ & & \\ & & \\ s & & c \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ & & \ddots & \vdots \\ & & & r_{nn} \\ a_1^{(0)} & a_2^{(0)} & \vdots & a_n^{(0)} \end{bmatrix} = \begin{bmatrix} \color{blue}{r_{11}^{+}} & \color{blue}{r_{12}^{+}} & \cdots & \color{blue}{r_{1n}^{+}} \\ & r_{22} & \cdots & r_{2n} \\ & & \ddots & \vdots \\ & & & r_{nn} \\ \color{blue}{0} & \color{blue}{a_2^{(1)}} & \vdots & \color{blue}{a_n^{(1)}} \end{bmatrix}, \quad \text{(A2)}$$

where the modified entries are denoted in blue, and $A \doteq \begin{bmatrix} a_1^{(0)} & \cdots & a_n^{(0)} \end{bmatrix}$. Since we consider unary observation models, and without loss of generality, we arbitrarily assume the first state is measured, i.e.:

$$a_1^{(0)} \equiv a \quad \text{and} \quad a_j^{(0)} = 0 \quad , \quad j > 1. \tag{A3}$$

We use the superscript to represent how many Givens rotations have been performed thus far. It is not difficult to show that $s \doteq -\frac{a_1^{(0)}}{r_{11}^{+}}$ and $c \doteq \frac{r_{11}}{r_{11}^{+}}$, and hence:

$$\left(r_{11}^{+}\right)^2 = r_{11}^2 + \left(a_1^{(0)}\right)^2 \quad , \quad a_j^{(1)} = \frac{-a_1^{(0)} r_{1j} + r_{11} a_j^{(0)}}{r_{11}^{+}} \tag{A4}$$

for $1 < j \leq n$. Note that although $a_j^{(0)} = 0$ for all j, $a_j^{(1)} \neq 0$. However, all such entries are *proportional* to a:

$$a_j^{(1)} = -a \frac{r_{1j}}{r_{11}^{+}}. \tag{A5}$$

This fact will be used in the sequel.

Consider now the ith application of Givens rotation, that nullifies $a_i^{(i-1)}$ and modifies additional entries as shown below.

$$\begin{bmatrix} r^+_{11} & \cdots & r^+_{1,i-1} & r^+_{1,i} & r^+_{1,i+1} & \cdots & r^+_{1n} \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & r^+_{i-1,i-1} & r^+_{i-1,i} & r^+_{i-1,i+1} & \cdots & r^+_{i-1,n} \\ & & & r_{ii} & r_{i,i+1} & \cdots & r_{in} \\ & & & & r_{i+1,i+1} & \cdots & r_{i+1,n} \\ & & & & & \ddots & \vdots \\ & & & & & & r_{nn} \\ 0 & \cdots & 0 & a_i^{(i-1)} & a_{i+1}^{(i-1)} & \cdots & a_n^{(i-1)} \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} r^+_{11} & \cdots & r^+_{1,i-1} & r^+_{1,i} & r^+_{1,i+1} & \cdots & r^+_{1n} \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & r^+_{i-1,i-1} & r^+_{i-1,i} & r^+_{i-1,i+1} & \cdots & r^+_{i-1,n} \\ & & & r^+_{ii} & r^+_{i,i+1} & \cdots & r^+_{in} \\ & & & & r_{i+1,i+1} & \cdots & r_{i+1,n} \\ & & & & & \ddots & \vdots \\ & & & & & & r_{nn} \\ 0 & \cdots & 0 & 0 & a_{i+1}^{(i)} & \cdots & a_n^{(i)} \end{bmatrix} \tag{A6}$$

The following expressions for r^+_{ii} and $a_j^{(i)}$ can be obtained by generalizing Eq. (A4):

$$\left(r^+_{ii}\right)^2 = r^2_{ii} + \left(a_i^{(i-1)}\right)^2 \quad , \quad a_j^{(i)} = \frac{-a_i^{(i-1)} r_{ij} + r_{ii} a_j^{(i-1)}}{r^+_{ii}}, \tag{A7}$$

with $i < j \leq n$, and, of course, $a_i^{(i)} = 0$. Recall that

$$\left|\Lambda^+\right| = \prod_{i=1}^{n} \left(r^+_{i,i}\right)^2 . \tag{A8}$$

We now prove Lemma 1 using mathematical induction.

Basis We show Lemma 1 holds for $n = 2$ (and $n = 1$). Using Eq. (A7), $|\Lambda^+|$ can be written as

$$\left|\Lambda^+\right| = \left(r^+_{1,1}\right)^2 \left(r^+_{2,2}\right)^2 = \left(r^+_{11}\right)^2 \left[r^2_{22} + \left(a_2^{(1)}\right)^2\right] = \left(r^+_{11}\right)^2 \frac{r^2_{22}\left(r^+_{11}\right)^2 + \left(-a_1^{(0)} r_{12} + r_{11} a_2^{(0)}\right)^2}{\left(r^+_{11}\right)^2} \tag{A9}$$

Recalling Eq. (A3) we get

$$\left|\Lambda^+\right| = r^2_{11} r^2_{22} + a^2 \left(r^2_{22} + r^2_{12}\right) = \eta_2 + a^2 \gamma_2 \left(R\right) \tag{A10}$$

with $\gamma_2(R) > 0$.

Inductive step Consider Lemma 1 holds for $n = k$:

$$\left|\Lambda^{+}\right| = \prod_{i=1}^{n=k} \left(r_{i,i}^{+}\right)^{2} = \eta_k + a^2 \gamma_k \left(R\right), \tag{A11}$$

and $\gamma_k\left(R\right) > 0$. Letting $\left|\Lambda_k^{+}\right| \doteq \prod_{i=1}^{k} \left(r_{i,i}^{+}\right)^{2}$, Eq. (A11) corresponds to

$$\left(r_{kk}^{+}\right)^{2} = \frac{1}{\left|\Lambda_{k-1}^{+}\right|} \left[\eta_k + a^2 \gamma_k \left(R\right)\right]. \tag{A12}$$

We now prove Lemma 1 is satisfied also for $n = k+1$, i.e.:

$$\left|\Lambda_{k+1}^{+}\right| = \eta_{k+1} + a^2 \gamma_{k+1}\left(R\right). \tag{A13}$$

We start with deriving an expression for $\left(r_{k+1,k+1}^{+}\right)^{2} = r_{k+1,k+1}^{2} + \left(a_{k+1}^{(k)}\right)^{2}$:

$$\left(r_{k+1,k+1}^{+}\right)^{2} = r_{k+1,k+1}^{2} + \frac{\left(-a_k^{(k-1)} r_{k,k+1} + r_{k,k} a_{k+1}^{(k-1)}\right)^{2}}{\left(r_{k,k}^{+}\right)^{2}} = \frac{1}{\left(r_{k,k}^{+}\right)^{2}} \left[r_{k+1,k+1}^{2} \left(r_{k,k}^{+}\right)^{2} + \left(-a_k^{(k-1)} r_{k,k+1} + r_{k,k} a_{k+1}^{(k-1)}\right)^{2}\right] \tag{A14}$$

Plugging in the expression for $\left(r_{kk}^{+}\right)^{2}$ from Eq. (A12) yields

$$\left(r_{k+1,k+1}^{+}\right)^{2} = \frac{1}{\left(r_{k,k}^{+}\right)^{2}} \left[\frac{r_{k+1,k+1}^{2}}{\left|\Lambda_{k-1}^{+}\right|} \left[\eta_k + a^2 \gamma_k \left(R\right)\right] + \left(-a_k^{(k-1)} r_{k,k+1} + r_{k,k} a_{k+1}^{(k-1)}\right)^{2}\right] \tag{A15}$$

Now, it is not difficult to show that by recursively using the relations (A7) we can express $a_k^{(k-1)}$ in terms of $a_k^{(1)}$. Recalling Eq. (A5) we get

$$a_k^{(k-1)} = \frac{-a_{k-1}^{(k-2)} r_{k-1,k} + r_{k-1,k-1} a_k^{(k-2)}}{r_{k-1,k-1}^{+}} = \cdots = a \frac{f_k^{(k-1)}(R)}{\sqrt{\left|\Lambda_{k-1}^{+}\right|}}, \tag{A16}$$

and similarly for $a_{k+1}^{(k-1)}$

$$a_{k+1}^{(k-1)} = \cdots = a \frac{f_{k+1}^{(k-1)}\left(R\right)}{\sqrt{\left|\Lambda_{k-1}^{+}\right|}}, \tag{A17}$$

where $f_k^{(k-1)}\left(R\right)$ and $f_{k+1}^{(k-1)}\left(R\right)$ are only functions of entries of R. We stress that this statement is valid only because of Eq. (A5), which corresponds to assuming unary measurement models.

Substituting expressions (A16) and (A17) into Eq. (A15) yields

$$\left(r^{+}_{k+1,k+1}\right)^{2}=\frac{1}{\left|\Lambda^{+}_{k}\right|}\left[\eta_{k+1}+a^{2}\gamma_{k+1}\left(R\right)\right], \tag{A18}$$

with

$$\gamma_{k+1}\left(R\right)\doteq r^{2}_{k+1,k+1}\gamma_{k}\left(R\right)+\left(-f^{(k-1)}_{k}\left(R\right)r_{k,k+1}+f^{(k-1)}_{k+1}\left(R\right)r_{k,k}\right)^{2}. \tag{A19}$$

Since $\left|\Lambda^{+}_{k+1}\right|=\left|\Lambda^{+}_{k}\right|\cdot\left(r^{+}_{k+1,k+1}\right)^{2}$, we get

$$\left|\Lambda^{+}_{k+1}\right|=\eta_{k+1}+a^{2}\gamma_{k+1}\left(R\right). \tag{A20}$$

We showed Lemma 1 holds for both the basis and inductive steps; hence, according to mathematical induction it holds for all natural n. ■

Appendix B: Proof of Lemma 2

Recall Eq. (10): $|\Lambda^{+}_{c}|=\prod_{i=1}^{n}\left(r^{+}_{c,ii}\right)^{2}$. Since R_c is diagonal, it is not difficult to show that (see Eq. (A7)) $r^{+}_{c,ii}=r_{c,ii}$ for $i>1$, i.e. only the upper left entry in matrix R_c is actually updated due to Jacobian A. Thus, we can write:

$$\left|\Lambda^{+}_{c}\right|=\left(r^{2}_{c,11}+a^{2}\right)\prod_{i=2}^{n}r^{2}_{c,ii}. \tag{B21}$$

According to Eq. (13), $r^{2}_{c,ii}=w_i\Sigma^{-1}_{ii}$ where Σ_{ii} is the corresponding entry on the diagonal of the covariance matrix $\Sigma\equiv\Lambda^{-1}$. Assuming, for simplicity $w_i=w=1/n$, Eq. (B21) turns into

$$\left|\Lambda^{+}_{c}\right|=\left(\Sigma^{-1}_{11}+na^{2}\right)n^{-n}\prod_{i=2}^{n}\Sigma^{-1}_{ii}. \tag{B22}$$

In practice, as detailed in [2] (see also [3]), calculation of Σ_{ii} can be efficiently performed directly from the nonzero entries of R, without the need in calculating an inverse of a large matrix:

$$\Sigma_{ll}=\frac{1}{r_{ll}}\left(\frac{1}{r_{ll}}-\sum_{j=l+1}^{n}r_{lj}\Sigma_{jl}\right) \tag{B23}$$

$$\Sigma_{il}=\frac{1}{r_{ii}}\left(-\sum_{j=i+1}^{l}r_{ij}\Sigma_{jl}-\sum_{j=l+1}^{n}r_{ij}\Sigma_{lj}\right) \tag{B24}$$

Based on Eqs. (B23)-(B24), it is possible to show that Σ_{ii} can be written, for all i, as

$$\Sigma_{ii}=\frac{\tilde{\gamma}_i}{\prod_{j=i}^{n}r^{2}_{jj}}, \tag{B25}$$

where $\tilde{\gamma}$ is only a function of elements of R. Recalling the definition of η_i from Eq. (15), we can write

$$\prod_{j=i}^{n} r_{jj}^2 = \prod_{j=1}^{n} r_{jj}^2 \Big/ \prod_{j=1}^{i-1} r_{jj}^2 = \frac{\eta_n}{\eta_{i-1}}, \tag{B26}$$

with the convention that $\eta_0 \doteq 1$. Denoting, for convenience $\gamma_{c,n-i+1} \doteq \tilde{\gamma}_i$, Eq. (B25) can be rewritten as

$$\Sigma_{ii} = \gamma_{c,n-i+1} \frac{\eta_{i-1}}{\eta_n}. \tag{B27}$$

Substituting Eq. (B27) into Eq. (B22) results in

$$\left|\Lambda_c^+\right| = \left(\eta_n + na^2\gamma_{c,n}\right) n^{-n} \frac{\prod_{i=2}^{n} \frac{\eta_n}{\eta_{i-1}}}{\prod_{i=1}^{n} \gamma_{c,n-i+1}}. \tag{B28}$$

Taking a closer look at $\prod_{i=2}^{n} \frac{\eta_n}{\eta_{i-1}}$ we can see that

$$\begin{aligned} \prod_{i=2}^{n} \frac{\eta_n}{\eta_{i-1}} &= \left(r_{22}^2 r_{33}^2 \cdots r_{nn}^2\right)\left(r_{33}^2 \cdots r_{nn}^2\right) \cdots \left(r_{n-1,n-1}^2 r_{nn}^2\right) r_{nn}^2 \\ &= \prod_{i=2}^{n} r_{ii}^{2(i-1)} \equiv \alpha_n / n^{-n}, \end{aligned} \tag{B29}$$

where α_n is defined in Eq. (17). Defining β_n as

$$\beta_n \doteq \prod_{i=1}^{n} \gamma_{c,n-i+1}, \tag{B30}$$

Eq. (B28) can be finally rewritten as

$$\left|\Lambda_c^+\right| = \frac{\alpha_n}{\beta_n} \left[\eta_n + na^2\gamma_{c,n}\right]. \tag{B31}$$

This completes the proof of Lemma 2. ■

Appendix C

In this appendix, we prove the relation from Conjecture 2

$$\left|\Lambda_c^+\right| = \frac{\alpha_n}{\beta_n} \left[n\left|\Lambda^+\right| - (n-1)\,\eta_n\right], \tag{C32}$$

for $n = 2$ and $n = 3$, thereby proving Conjectures 1 and 2 for these cases. In the following we use, for simplicity, $w_i = w = 1/n$.

We start with $n = 2$. According to Lemma 1 it is possible to show that

$$\left|\Lambda^{+}\right| = \prod_{i=1}^{2}\left(r_{ii}^{+}\right)^{2} = a_{1}^{2}\left(r_{12}^{2}+r_{22}^{2}\right)+r_{11}^{2}r_{22}^{2}. \tag{C33}$$

On the other hand, using Eqs. (B23)-(B24) we get

$$r_{c11}^{2} = \frac{1}{2}\frac{r_{11}^{2}r_{22}^{2}}{r_{12}^{2}+r_{22}^{2}} \quad , \quad r_{c22}^{2} = \frac{1}{2}r_{22}^{2}. \tag{C34}$$

Recalling Eq. (B21), $|\Lambda_c^{+}| = \left(r_{c11}^{2}+a_{1}^{2}\right)r_{c22}^{2}$, and substituting Eq. (C34) we obtain the following recursive relation in $|\Lambda^{+}|$:

$$\left|\Lambda_c^{+}\right| = \frac{1}{2^{2}}r_{22}^{2}\frac{2\left|\Lambda^{+}\right|-r_{11}^{2}r_{22}^{2}}{r_{12}^{2}+r_{22}^{2}}. \tag{C35}$$

This expression indeed corresponds to Eq. (C32).

Considering now $n = 3$, it is possible to show that

$$\left|\Lambda^{+}\right| = r_{11}^{2}r_{22}^{2}r_{33}^{2}+a_{1}^{2}\left[r_{33}^{2}\left(r_{12}^{2}+r_{22}^{2}\right)+\left(r_{12}r_{23}-r_{13}r_{22}\right)^{2}\right]. \tag{C36}$$

Similarly, to the previous case, using Eqs. (B23)-(B24) we get $r_{c22}^{2} = \frac{1}{3}\frac{r_{22}^{2}r_{33}^{2}}{r_{23}^{2}+r_{33}^{2}}$, $r_{c33}^{2} = \frac{1}{3}r_{33}^{2}$ and

$$r_{c11}^{2} = \frac{1}{3}\frac{r_{11}^{2}r_{22}^{2}r_{33}^{2}}{r_{33}^{2}\left(r_{12}^{2}+r_{22}^{2}\right)+\left(r_{12}r_{23}-r_{13}r_{22}\right)^{2}}. \tag{C37}$$

Noting that $|\Lambda_c^{+}| = \left(r_{c11}^{2}+a_{1}^{2}\right)r_{c22}^{2}r_{c33}^{2}$ and substituting the above relations it is not difficult to show that

$$\left|\Lambda_c^{+}\right| = \frac{1}{3^{3}}\frac{r_{22}^{2}r_{33}^{4}}{r_{23}^{2}+r_{33}^{2}}\left(\frac{3\left|\Lambda^{+}\right|-2r_{11}^{2}r_{22}^{2}r_{33}^{2}}{r_{33}^{2}\left(r_{12}^{2}+r_{22}^{2}\right)+\left(r_{12}r_{23}-r_{13}r_{22}\right)^{2}}\right). \tag{C38}$$

As previously, this expression indeed corresponds to Eq. (C32).

Appendix D: Additional Results

In this appendix we demonstrate the proposed concept in a larger sensor-deployment problem than the one considered in [1]. Specifically, we have a 40×40 grid (instead of 10×10 as in [1]), which corresponds to $X \in \mathbb{R}^{1600}$ and $\Sigma \in \mathbb{R}^{1600\times1600}$. Figure 1 provides the results: the prior uncertainty field is shown in Figure 1a, the corresponding running time for making 10 greedy sensor deployment decisions using the original and conservative information space is shown in Figure 1b. Figures (1c)-(1f) show the impact of candidate actions is preserved, as stated by Conjecture 3. While this is not easily inferred from

Figure 1c, we provide a zoom-in in Figure 1d. We also show in Figures 1e and 1f the impact of candidate actions when sorting the x-axis (i.e. candidate actions) considering the objective function[1] $log(det(\Lambda^{+}))$ that uses the original information space. The same ordering is then used for the conservative information space. Thus, if the trend was different, the resulting curve for the conservative case would not be monotonically decreasing.

References

[1] V. Indelman, "No correlations involved: Decision making under uncertainty in a conservative sparse information space," in *IEEE Robotics and Automation Letters (RA-L)*, accepted.

[2] G. Golub and R. Plemmons, "Large-scale geodetic least-squares adjustment by dissection and orthogonal decomposition," *Linear Algebra and Its Applications*, vol. 34, pp. 3–28, Dec 1980.

[3] M. Kaess and F. Dellaert, "Covariance recovery from a square root information matrix for data association," *Robotics and Autonomous Systems*, 2009.

[1] We use the logarithm, a monotonic function, for numerical reasons.

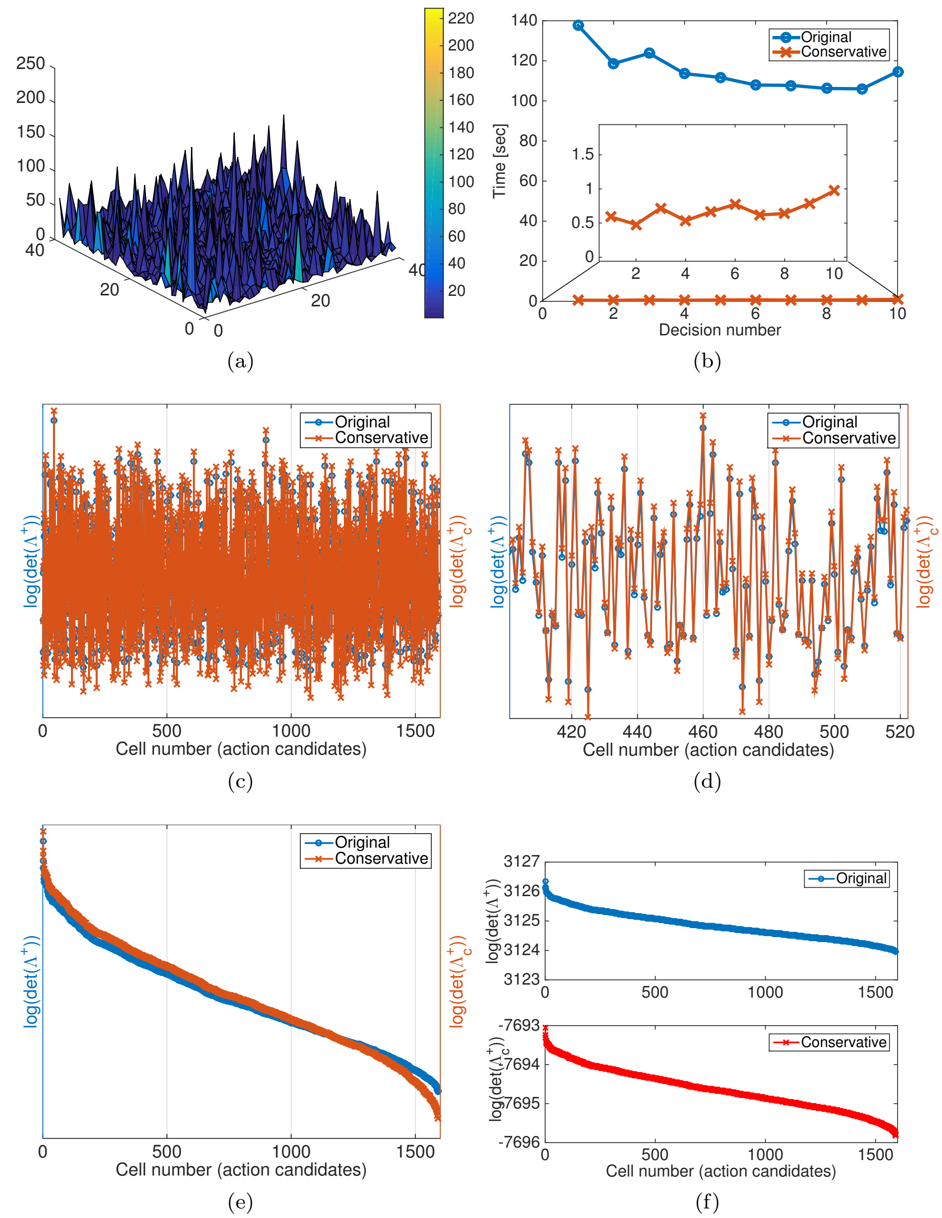

Figure 1: Uncertainty field synthetic example: (a) A priori variance in each cell of the $N \times N$ grid with $N = 40$, which corresponds to $X \in \mathbb{R}^{1600}$ and $\Sigma \in \mathbb{R}^{1600 \times 1600}$; (b) Timing results for making 10 sequential greedy decisions using the original and conservative information space. (c) Impact of each candidate decision (sensor location) using the original and conservative information matrices. A zoom-in is shown in (d). Although values are different, the trend is identical in both cases for any two candidate actions, as stated by Conjecture 3. (e) Impact of candidate decisions from (c), with both curves sorted according to $log(det(\Lambda^{+}))$. The monotonically decreasing curve for $log(det(\Lambda_c^{+}))$ indicates an identical trend in both cases. (d) Numerical values of each curve from (e).