Topological Multi-Robot Belief Space Planning in Unknown Environments*

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Abstract— **In this paper we introduce a novel concept, topological belief space planning (BSP), that uses topological properties of the underlying factor graph representation of future posterior beliefs to direct the search for an optimal solution. This concept deviates from state-of-the-art BSP approaches and is motivated by recent results which indicated, in the context of graph pruning, that topological properties of factor graphs dominantly determine the estimation accuracy. Topological space is also often less dimensional than the embedded state space. In particular, we show how this novel concept can be used in multi-robot belief space planning in high-dimensional state spaces to overcome drawbacks of state-of-the-art approaches: computational intractability of an exhaustive objective evaluation for all candidate path combinations from different robots and dependence on the initial guess in the announced path approach, which can lead to a local minimum of the objective function. We demonstrate our approach in a synthetic simulation.**

I. Introduction

Planning under uncertainty or Belief Space Planning (BSP) can be formulated as a Partially Observed Markov Decision Process (POMDP) [17] defined over the space of probability distributions of the state space, also referred to as the belief space. The goal of multi-robot BSP is to determine optimal control actions for all robots in a group over the belief space according to a specific task-related objective. In collaborative estimation, sensor deployment, multi-robot tracking, an objective is often defined as an information-theoretic cost over the variables of interest (e.g. minimizing the state uncertainty). On the other hand, multi-robot search and rescue applications might require fast coverage of the environment and high mapping accuracy, and collaborative robotic manipulators highly accurate pose of their end effectors to avoid potential collisions and low control effort to produce smooth motions.

In the context of mobile robots navigation and planning, this problem is found in active SLAM in which a robot, while operating in an unknown or uncertain environment, has to autonomously decide its future motion and estimate its own pose and the state of the environment simultaneously using only on-board sensors (see e.g. [29], [16], [15], [9]).

Finding globally optimal solutions to the POMDP problem is computationally intractable, even with finite planning horizons, discrete states, actions, and observations [22]. In particular, solving exactly the decentralized POMDP has been shown to be nondeterministic exponential (NEXP) complete [3]. As a result, a large subset of prior work has focused on approximately solving the POMDP problem for Gaussian belief spaces using sampling-based motion planners (e.g. [11], [19], [25], [1]) or optimization-based methods (e.g. [9], [24], [30]).

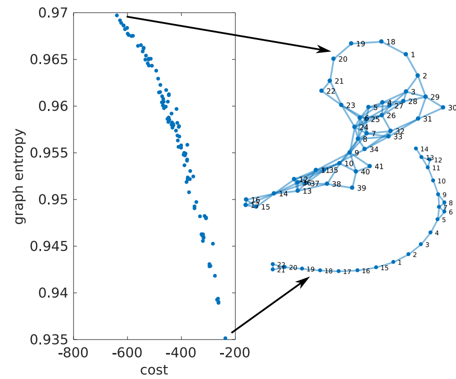

Fig. 1: Concept illustration. Information-theoretic cost in BSP is mainly influenced by topology of the underlying factor graphs represented by simple undirected graphs. More complex graphs tend to have higher entropy and produce more accurate estimates. Each point in the plot corresponds to a multi-robot posterior belief associated with a single variation of robots' control actions.

In a decentralized discrete BSP framework, each robot r in a group of R robots generates a finite number of candidate paths by sampling the environment among which it has to determine the best one according to a specified objective function. Let us assume for simplicity that each robot generates N candidate paths. Determining the optimal controls involves considering *all* path combinations between different robots (N^R combinations), which is computationally intractable for large number of candidate paths and high-dimensional state spaces since evaluating the objective function for each combination involves multi-robot inference.

Instead, a common (sub-optimal) approach for decentralized belief space planning is for each robot r to consider only its own candidate paths and the *announced* paths of other robots, see e.g. [20], [2]. The robot can then select the best path, according to the objective function, and announce this path to other robots, which then repeat the same procedure on their end. Such an approach reduces the exponential complexity in number of robots to a linear complexity, and can be viewed as a decentralized coordindated descent [20], [2], i.e. where robots either repeat this process until convergence [2] or at some frequency [20]. Performance guarantees of such an approach are analyzed in [2]. In [27] an approach is developed to identify and efficiently re-evaluate, while re-

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using calculations, only impacted candidate paths due to an update in the announced path. However, this strategy still requires $R \cdot N \cdot N_{iter}$ objective function evaluations, where N_{iter} is the number of iterations until all robots reach an agreement and do not change their decisions any more. The other problem of the announced paths approach is that it can be stuck in local minimum depending on the initialization.

As our first contribution, we propose a novel concept, named *topological belief space planning*, that uses topological properties of the underlying factor graph representation of future posterior beliefs to direct the search for an optimal solution. Additionally, we show in this paper how this general concept can be applied to multi-robot decentralized BSP in high dimensional state spaces to reduce its computational complexity. In multi-robot BSP we sub-sample the set of robots' candidate actions in a topological space. In our approach, each multi-robot belief that corresponds to a certain combination of robots' actions, is represented by a factor graph to which we assign a signature in a topological space. Signatures are ranked, and then sub-sampled and the objective function is only evaluated on those samples, rather than on all N^R action combinations, yet with the accuracy compared to exhaustive (optimal) approach. This main idea is illustrated in Figure 1.

Topological representations in SLAM have been studied mostly through higher-level representations of the environment. In [28], an algorithm for multi-robot topological description and exploration of unknown environments is presented. Homology classes of trajectories are identified and used to distribute the task of exploration among different groups of robots when confronted by obstacles. A systematic algorithm for probabilistic topological mapping of an environment from a sequence of sensor-independent measurements is proposed in [26].

Only recently, an impact of the topology of the underlaying estimation problem has been given more attention. In their seminal work on the impact of graph connectivity on the reliability of SLAM [13], the authors give bounds of the maximum likelihood estimation error of the pose SLAM. They show that D-optimal design is the one that maximizes the number of spanning trees (*tree-connectivity*) of the corresponding SLAM graph in special classes of the pose SLAM problem, with linear measurement model, like those arising in sensor networks and 2D/3D SLAM with a known robot's orientation, and isotropic noise. Furthermore, it was demonstrated that in general non-linear SLAM, determinant of the information matrix is strongly positively correlated with tree-connectivity of the graph. Analytical expressions are given for finding the bounds of an error of such approximation that depends on the distances between graph nodes and the noise variances. The authors show how this metric can be used also in solving a problem of measurement selection and graph pruning. Although sparsity of the SLAM structure can be exploited in calculating the number of spanning trees, it still depends on the state dimension.

Our proposed concept of topological BSP is motivated by the above work but addresses a BSP problem. This problem is different and more general than graph-pruning and measurement selection problems described above, since it considers multiple path realizations from different controls, with greater variety in other non-topological factors that influence estimation accuracy, e.g non-fixed geometry and different path lengths.

As our second contribution in this work, we propose a new topology metric, Von Neumann entropy ([21], [23]), to be used in topological BSP as graph signature, and show empirically how it is related to D-optimality criterion. This metric depends on the whole graph spectrum (all eignevalues) and can be calculated efficiently without depending on the state dimension by approximating it with a function of graph node degrees which is especially important when operating in high dimensional state space as in a multi-robot BSP problem.

II. Probabilistic Formulation and Notations

Decentralized multi-robot belief space planning (BSP) considers a group of R robots operating in unknown or uncertain environment, aiming to autonomously decide their future actions based on information accumulated thus far and given an user defined objective function J. In this work, we further assume that each robot can obtain full information about the state of the multi-robot system in every planning step. This makes the inference process centralized while only the planning is decentralized.

Let $\mathbb{P}(X_k^r|\mathcal{Z}_{0:k}^r, \mathcal{U}_{0:k-1}^r)$ represent the posterior probability density function (pdf) at planning time t_k over states of interest X_k^r of robot r. In this work, for simplicity we assume pose SLAM framework where states of interest are robot's current and past poses, i.e. $X_k^r = \{x_0^r, x_1^r, \ldots, x_k^r\}$. $\mathcal{Z}_{0:k}^r$ and $\mathcal{U}_{0:k-1}^r$ denote, respectively, all observations and controls by time t_k. Consider conventional state transition and observation models

$$x_{i+1} = f(x_i, u_i, w_i) \ , \ z_{i,j} = h(x_i, x_j, v_{i,j}) \tag{1}$$

with zero-mean Gaussian process and measurement noise $w_i \sim N(0, \Omega_w)$ and $v_{i,j} \sim N(0, \Omega_{vij})$, and with known information matrices Ω_w and Ω_{vij}. Denoting the corresponding probabilistic terms to Eq. (1) by $\mathbb{P}(x_i|x_{i-1}, u_{i-1})$ and $\mathbb{P}(z_{i,j}|x_i, x_j)$, the pdf $\mathbb{P}(X_k^r|\mathcal{Z}_{0:k}^r, \mathcal{U}_{0:k-1}^r)$ can be written as

$$\mathbb{P}(X_k^r|\mathcal{H}_k^r) \propto \mathbb{P}(x_0^r) \prod_{i=1}^{k} \mathbb{P}(x_i^r|x_{i-1}^r, u_{i-1}^r) p(Z_i^r|X_i^r) \tag{2}$$

where the history $\mathcal{H}_k^r$ is defined as $\mathcal{H}_k^r \doteq \{\mathcal{Z}_{0:k}^r, \mathcal{U}_{0:k-1}^r\}$.

The measurement likelihood term $\mathbb{P}(Z_i^r|X_i^r)$ can be expanded in terms of individual observations, $\mathbb{P}(Z_i^r|X_i^r) = \prod_{j=1}^{n_i} \mathbb{P}(z_{i,j}^r|X_{i,j}^r)$. Here, $Z_i^r \doteq \{z_{i,j}^r\}_{j=1}^{n_i}$ and n_i denotes the number of observations acquired at time t_i and $X_{i,j}^r \subseteq X_i^r$ represents involved variables in the jth observation model. In the case of pose SLAM, $z_{i,j}^r$ is a relative pose constraint between poses x_i^r and x_j^r.

Now, consider all the R robots in the group. Let $\mathbb{P}(X_k|\mathcal{H}_k)$ represent the pdf over the joint state X_k at time t_k, where $X_k \doteq \{X_k^r\}_{r=1}^R$ and $\mathcal{H}_k \doteq \{\mathcal{Z}_{0:k}, \mathcal{U}_{0:k-1}\}$, with $\mathcal{Z}_{0:k} \doteq \{\mathcal{Z}_{0:k}^r\}_{r=1}^R$ and $\mathcal{U}_{0:k-1} \doteq \{\mathcal{U}_{0:k-1}^r\}_{r=1}^R$.

In a decentralized multi-robot framework, each robot maintains the joint belief $\mathbb{P}(X_k|\mathcal{H}_k)$ on its own while communicating to each other relevant pieces of information. We assume, for simplicity, each robot is capable of calculating the joint pdf at planning time $\mathbb{P}(X_k|\mathcal{H}_k)$ using one of the recently developed approaches (e.g. [5], [10]). We note that given transition and observation models (1), it is sufficient for each robot to only transmit (in addition to what is required by multi-robot inference) its own control actions. Any other robot that receives this information can then formulate the multi-robot belief $\mathbb{P}(X_k|\mathcal{H}_k)$ [20].

Let a user-defined objective function be $J(\mathcal{U}) = \mathbb{E}\left[c_l(b[X_{k+L}])\right]$, where the expectation is taken with respect to future observations of all robots, and where c_l represents a cost function of the joint belief $b[X_{k+L}]$ (to be defined) at the end of the planning horizon. For simplicity, we use the same planning horizon L for all robots. In this paper we aim to minimize an entropy of the multi-robot belief. The objective function J for multivariate Gaussian belief is therefore:

$$J(\mathcal{U}) = \frac{n}{2}\ln(2\pi\mathrm{e}) + \frac{1}{2}\ln|\Sigma(X_{k+L})|, \tag{3}$$

where $\Sigma(X_{k+L})$ denotes estimated covariance of the multi-robot belief $b[X_{k+L}]$, and n dimensionality of the joint state X_{k+L}. Notice that minimizing the global entropy (3) corresponds to maximizing the total information gain obtained by robots executing the same trajectories. Such a form naturally supports *collaborative active* state estimation, where each robot aims to improve estimation accuracy of the joint state.

Therefore, our objective is to find the optimal controls

$$\mathcal{U}^\star = \arg\min_{\mathcal{U}} J(\mathcal{U}) \tag{4}$$

for all robots in the group, considering a multi-robot decentralized discrete BSP framework, as described below.

A. Generating actions

Each robot r in every planning session at time t_k generates finite number (N_r) of candidate paths. One way to generate them is by using one of the existing sampling-based motion planning approaches (e.g. RRT, RRG, PRM). Every such path corresponds to one non-myopic control action. This means that a set $\mathcal{U}$ of all possible combinations of control actions of R different robots in the group contains $\prod_{r=1}^{R} N_r$ elements. Since the robots can have numerous candidate paths, determining the optimal controls (4) is computationally intractable in particular for high-dimensional state spaces considered herein since it requires multi-robot belief propagation for *all* controls.

B. Multi-robot inference

Considering for now only a single robot r, let us assume that future sampled states of robot r along one of its candidate paths generated at planning time t_k are $\{x^r_{k+1}, \ldots, x^r_{k+L}\}$. The corresponding joint belief over the entire path $\mathcal{P}^r$, is

$$b[\mathcal{P}^r] \doteq \mathbb{P}(X^r_k, x^r_{k+1}, \ldots, x^r_{k+L}|\mathcal{H}^r_k, U(\mathcal{P}^r), Z(\mathcal{P}^r)), \tag{5}$$

where $U^r(\mathcal{P}^r)$ and $Z^r(\mathcal{P}^r)$ represent, respectively, the corresponding controls and (unknown) observations to be acquired by following the path $\mathcal{P}^r$. This pdf can be explicitly written in terms of the belief at planning time and the corresponding state transition and observation models as (see Eq. (2))

$$\begin{aligned} b[\mathcal{P}^r] &= \mathbb{P}(X^r_k|\mathcal{H}^r_k)\mathbb{P}(\mathcal{P}^r|U^r(\mathcal{P}^r), Z^r(\mathcal{P}^r)) \\ &= \mathbb{P}(X^r_k|\mathcal{H}^r_k)\prod_{l=1}^{L(\mathcal{P}^r)} \mathbb{P}(x^r_{k+l}|x^r_{k+l-1}, u^r_{k+l-1})\mathbb{P}(Z^r_{k+l}|X^r_{k+l}). \end{aligned} \tag{6}$$

The measurement likelihood term $\mathbb{P}(Z^r_{k+l}|X^r_{k+l})$ can be further expanded, similarly to Eq. (2). Here, X^r_{k+l} is the joint state up to the lth future state from the current state along the path $\mathcal{P}^r$, i.e.:

$$X^r_{k+l} = X^r_{k+l}(\mathcal{P}^r) \equiv \mathcal{P}^r_{k+l} \doteq \{X^r_k, x^r_{k+1}, \ldots, x^r_{k+l}\}. \tag{7}$$

We now proceed to the multi-robot case and consider different paths $\mathcal{P}^r$ for each robot $r \in \{1, \ldots, R\}$. Letting $\mathcal{P} \doteq \{\mathcal{P}^r\}^R_{r=1}$, the multi-robot belief is given by

$$\begin{aligned} b[\mathcal{P}] \;=\; \mathbb{P}(X_k|\mathcal{H}_k)\prod_{r=1}^{R}\Bigg[\prod_{l=1}^{L(\mathcal{P}^r)} &\mathbb{P}(x^r_{k+l}|x^r_{k+l-1}, u^r_{k+l-1}) \\ &\cdot\mathbb{P}(Z^r_{k+l}|X^r_{k+l})\prod_{\{i,j\}}\mathbb{P}(z^{r,r'}_{i,j}|x^r_i, x^{r'}_j)\Bigg], \end{aligned} \tag{8}$$

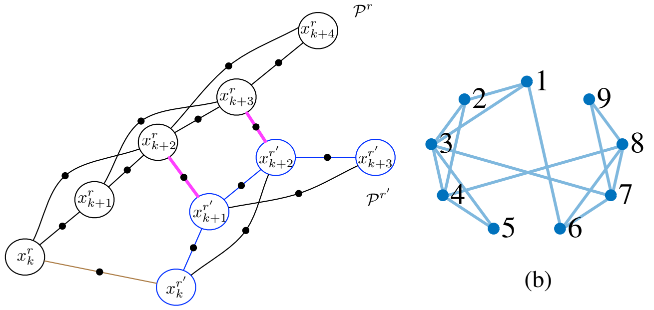
where the last product corresponds to multi-robot constraints that can involve different time instances, representing mutual observations of a scene. The ordered index set $\{i, j\}$ in Eq. (8) represents the time indices that facilitate multi-robot constraints among the robots $\{r, r'\}$. We assume a given criteria function $\mathbf{cr_{MR}}(x^r_i, x^{r'}_j)$ that determines if there should be a multi-robot constraint between the two vertices x^r_i and $x^{r'}_j$. This function is conceptually similar to the indicator function used in [20], while in our previous work [8] we used a simpler criteria (relative distance).

C. Topology of multi-robot belief

The joint belief (8) can be represented by a factor graph graphical model [18], as illustrated in Figure 2a. A factor graph (*FG*) is a bipartite graph whose nodes consist of factors F and variables V. The variables represent the random variables in the estimation problem, whereas the factors represent probabilistic information on those variables, derived from measurements or prior knowledge. Different candidate paths $\mathcal{P}$ typically yield different factor graphs.

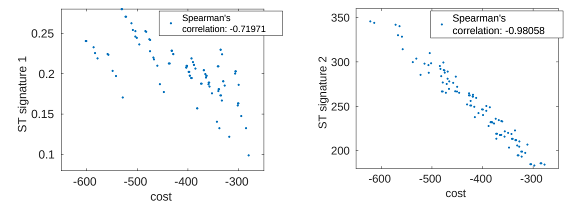
Now, let us consider a topology induced by *FG* that represents the multi-robot belief (8). In the case F contains only binary factors, topology of the *FG* can be further simplified in the following way.

Let $\mathcal{L}$ be a vertex labeling function of FG that assigns to its variables V a set of unique labels $\Gamma = \{1, 2, \ldots, |V|\}$. We define a topological graph $G \doteq (\Gamma, E)$ as a graph whose nodes are in the set Γ and whose edges are $E = \{(i, j), i, j \in \Gamma \Leftrightarrow$ a factor exists in F involving variables $\mathcal{L}^{-1}(i)$ and $\mathcal{L}^{-1}(j)\}$.



(a) (b)

Fig. 2: Graph representations of the multi-robot belief. **(a)** Factor graph; **(b)** Topological graph.



(a) (b)

Fig. 3: Signatures based on the number of spanning trees (ST) vs cost (3). **(a)** Normalized tree connectivity; **(b)** Adapted ST signature (15).

The corresponding topological graph to factor graph from Figure 2a is shown in Figure 2b.

In this paper, we study BSP structural properties based on the spectrum of the normalized Laplacian matrix $\hat{L}$ associated to the graph G. Laplacian matrix of a graph G is by definition $L(G) = D(G) - A(G)$, where $A(G)$ is its $|\Gamma| \times |\Gamma|$ adjacency matrix with elements

$$A(i,j) = \begin{cases} 1, \text{if } (i,j) \in E \\ 0, \text{ otherwise} \end{cases}$$

and $D(G)$ its node degree matrix defined as a diagonal matrix with graph node degrees on its main diagonal, i.e. $D(i,i) = d(i) = \sum_{j \in \Gamma} A(i,j)$. Finally, a normalized Laplacian matrix is defined as $\hat{L}(G) = D^{-1/2} L D^{-1/2}$ which can be also written in the form

$$\hat{L}(i,j) = \begin{cases} 1, \text{if } i = j, d(i) \neq 0 \\ \dfrac{-1}{\sqrt{d(i)d(j)}}, (i,j) \in E \\ 0, \text{ otherwise} \end{cases}$$

Let a spectral decomposition of the normalized Laplacian matrix be $\hat{L} = Q \Lambda Q^{-1}$, where $\Lambda = diag(\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_{|\Gamma|})$ is a diagonal matrix of the eigenvalues of $\hat{L}$, and Q is an orthogonal matrix whose columns are their corresponding eigenvectors. The normalized Laplacian matrix of G is symmetric and positive semi-definite so all its eigenvalues are real and non-negative. Hence, they can be ordered

$$\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \cdots \leq \hat{\lambda}_{|\Gamma|},$$

with $\hat{\lambda}_1 = 0$, $\hat{\lambda}_{|\Gamma|} \leq 2$, and $\sum_{i=1}^{|\Gamma|} \hat{\lambda}_i = |\Gamma|$ since G has no isolated nodes (for proof see [4]).

III. Approach

We propose a method that reduces the complexity of multi-robot decentralized belief space planning in high dimensional state spaces by sub-sampling the set of robots' candidate actions in a topological space. Each factor graph, resulted from a certain combination of robots' actions, is represented by a signature in a topological space. Signatures are clustered ranked, and then sub-sampled and objective is only evaluated on those samples (much less than exhaustive) yet with the accuracy compared to exhaustive (optimal) approach. Since we operate on the whole topological space of the resulting FGs, our method is not prone to local minima.

Our method, topological BSP, is summarized in Algorithm 1 and in the following we discuss its main parts.

Algorithm 1 Topological BSP

Input: set of factor graphs FG
Output: approximate solution to the BSP, $\hat{\mathcal{U}}$
1: represent each FG with a topological graph G
2: determine S_G, set of graph signatures of G
3: rank graphs according to their signatures
4: U = { top ranked candidates in S_G}
5: $\hat{\mathcal{U}}$ = arg min $J(U)$

A. Graph signature

The crucial idea in designing a graph signature is that it should reflect, in great extent, an optimization objective J, i.e. there should exist a strong correlation between a graph's topological signature and an objective. If such a signature exists, then it can be used to direct the search for an optimal solution. Another important property it should possess is that its computation is much faster than explicit evaluation of an objective function.

In this work, we propose a new metric of graph complexity for characterizing a graph G associated with a factor graph FG as signature candidate for the optimization objective defined by (3), the Von Neumann entropy of a graph.

The von Neumann entropy $H_{VN}(G)$ of graph G is the Shannon entropy associated with its normalized Laplacian's eigenvalues $\{\hat{\lambda}_i\}_{i=1}^{|\Gamma|}$ and was introduced in [23]

$$H_{\text{VN}}(G) = -\sum_{i=1}^{|\Gamma|} \frac{\hat{\lambda}_i}{|\Gamma|} \ln \frac{\hat{\lambda}_i}{|\Gamma|}. \quad (9)$$

Using Han's quadratic approximation [7], it can be simplified to

$$H_{VN}(G) \approx \sum_{i=1}^{|\Gamma|} \frac{\hat{\lambda}_i}{|\Gamma|} \left(1 - \frac{\hat{\lambda}_i}{|\Gamma|}\right) \quad (10)$$

$$= \frac{\sum_{i=1}^{|\Gamma|} \hat{\lambda}_i}{|\Gamma|} - \frac{\sum_{i=1}^{|\Gamma|} \hat{\lambda}_i^2}{|\Gamma|^2} = \frac{\text{Tr}[\hat{L}]}{|\Gamma|} - \frac{\text{Tr}[\hat{L}^2]}{|\Gamma|^2}. \quad (11)$$

The normalized Laplacian matrix $\hat{L}$ is symmetric with unit diagonal. Therefore, $\text{Tr}[\hat{L}] = |\Gamma|$, and, after some basic matrix manipulations, that we omit here for space reasons,

$$\text{Tr}[\hat{L}^2] = |\Gamma| + \sum_{(i,j) \in E} \frac{1}{d(i)d(j)}.$$

Putting all together, the final expression for the Von Neumann entropy approximation we use as a graph signature is

$$H_{VN}(G) \approx 1 - \frac{1}{|\Gamma|} - \frac{1}{|\Gamma|^2} \sum_{(i,j)\in E} \frac{1}{d(i)d(j)}. \tag{12}$$

Notice that its computation depends only on the degree matrix $D(G)$ and, in general case, has quadratic complexity in the number of nodes, $O(|\Gamma|^2)$ but in the case of BSP, where $\hat{L}$ is sparse it depends only on the small number of non-zero elements of $A(G)$, i.e. the number of edges $|E|$ in the graph G. Also, the expression (12) can be computed incrementally, as new edges (measurements) are added to the multi-robot factor graph as the robots explore the environment or re-plan their paths. We leave further investigation of this aspect to future research.

Here, we also analyzed the tree connectivity metric $\tau(G) \doteq \ln(t(G))$ proposed in [13] in the context of measurement selection and graph prunning. This metric is based on the number of spanning trees $t(G)$ in a graph G, and we show how it can be adapted for use in BSP for optimizing objective (3).

In [12], normalization of $\tau(G)$ was proposed to deal with different state dimensions of SLAM pose graphs, but this normalization does not give good signature candidate (we denote it ST signature 1) in BSP problem, see Fig. 3a. Instead, we derive another signature candidate, denoted as ST signature 2, based on $\tau(G)$ in the following way. According to [13]

$$\ln|\Sigma(X_{k+L})| \approx -3\ln(t(G)) - \ln(\eta), \tag{13}$$

where η depends on the noise variances and geometry and, in the case of BSP, on the state dimension n. We can write $\eta = C\xi$, where C captures the geometry and the noise parameters and is assumed constant, i.e. independent on the FG topology, while ξ is a function of state dimension n. Minimizing (3) can be therfore formulated as

$$\mathcal{U}^\star = \arg\max_{\mathcal{U}}[-J(\mathcal{U})] \approx$$
$$\arg\max_{\mathcal{U}} \left[\frac{3}{2}\ln t(G) + \frac{\ln(\xi)}{2} - \frac{n}{2}\ln(2\pi e) + const.\right]. \tag{14}$$

ξ can be determined from the odometry factor graph associated with $b[X_{k+L}]$. Given that process noise information matrix is Ω_w and its topological graph G^o contains n nodes, it follows from (13) since $t(G^o) = 1$, that $\ln(\eta) = n\ln(|\Omega_w|)$[1] Finally, it follows that, ST signature 2, the signature that maximizes (14), can be determined up to a constant as

$$\frac{3}{2}\tau(G) + \frac{n}{2}[\ln|\Omega_w| - \ln(2\pi e)], \tag{15}$$

and one can observe it better captures the correlation with the cost on the same set of candidate actions (Fig. 3b).

[1]Here we used a property that the determinant of a block diagonal matrix is the product of the determinants of its diagonal blocks and Lemma 3 from [14] under the same assumptions.

B. Sub-sampling and Optimization

After obtaining graph signatures for all the possible path combinations, we proceed by sub-sampling them (Alg. 1, step 4). The sub-sampling strategy depends on the optimization objective's relation to the signature and on the available time resources we have. If we can assume that the BSP solution has high correlation with the signature, e.g. minimizing uncertainty (3) corresponds to maximizing graph's complexity (12), we can then choose top N_s best ranked signatures, evaluate the objective function for the corresponding actions, and then select the best among them (Alg. 1, step 5). Another alternative we envision, is to cluster the signatures and search for the solution inside the best topology cluster. In that case, N_s is proportional to the number of clusters and we assume that similar topology classes will produce similar cost. Notice that the proposed topological BSP algorithm can be considered as anytime algorithm, as its solution quality depends on the the available computational budget and gets monotonically improved as more samples are considered.

IV. RESULTS

We evaluate our approach in a simulation of collaborative active SLAM involving two robots operating in unknown and GPS-deprived environment. We study empirically different signature candidates and their correspondences to information theoretic cost, D-optimality criterion (3). The first one is based on the number of spanning trees in a topological graph (ST), and the second, on the graph entropy newly introduced in this paper in the context of BSP, the von Neumann entropy (VN). We also analyze convergence rate of the proposed algorithm and state-of-the-art non-topological multi robot BSP algorithms, announced path (AP) and exhaustive search (EX), with respect to the number of objective function evaluations.

In this basic evaluation we use a prototype implementation in Matlab using GTSAM [6] to investigate key aspects of the proposed approach. A probabilistic roadmap (PRM) [11] is used to discretize the environment and generate candidate paths over the roadmap. Figure 5g shows the considered scenario for two robots and the generated 10 candidate paths for each robot in a single planning session.

In the first experiment, we ran 25 planning sessions per every considered algorithm, each time generating randomly candidate paths of the robots in the same scenario (fixed start and goal positions, the process and the measurement noise). Given two robots, each having 10 candidate paths, means that in every planning session 100 different factor graphs are formed representing the multi-robot belief of the corresponding controls and predicted observations. Similar to [27], we use a simple heuristic for the function $\mathbf{cr_{MR}}(x_i^r, x_j^r)$ to determine if two poses admit a multi-robot constraint: these constraints, possibly involving different future time instances, are formulated between any two poses with relative distance closer than $d = 320$ meters. In every planning session, topological BSP algorithms choose *one* sample (path combination) to evaluate the objective function according to

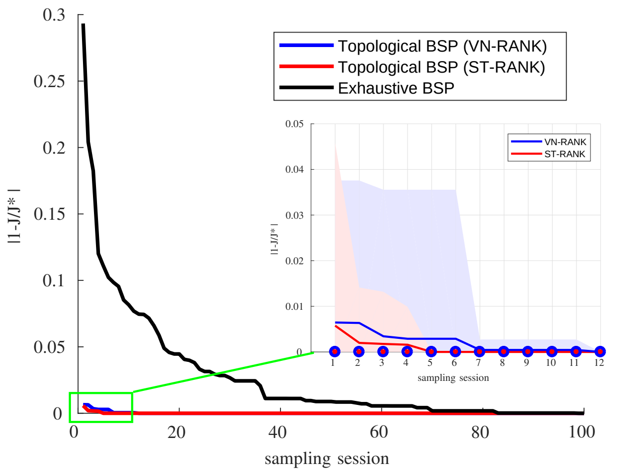


Fig. 4: Convergence rate of the relative error. In the zoomed in region we show its statistics. Solid lines represent a mean of the relative error, circles its median and shaded areas 95% confidence region.

their corresponding signature and sub-sampling strategy. For the objective of minimizing the global entropy, we select the top best ranked signature. We denote these approaches VN-RANK and ST-RANK. In Fig. 4, we show performance of VN-RANK, ST-RANK and exhaustive approach after each sampling session in terms of an average relative error (RE) defined as

$$RE \doteq |1 - J/J^*|. \tag{16}$$

Here, $J^* \doteq J(\mathcal{U}^\star)$, while J represents objective function value for the best action (path combination) sampled thus far. Exhaustive approach randomly selects the candidates until it evaluates all of them. We confirm that topological properties are indeed an effective discriminant of the accuracy of BSP since both VN-RANK and ST-RANK converge much faster than random sampling. Also, ST-RANK and VN-RANK have similar performance which justifies the use of VN-RANK as the graph signature since its computation can be done much faster for high dimensional state spaces due to its dependence only on the graph node degrees. To compare to the announced path (AP) approach, we measured the number of objective function evaluations until convergence. In the case of VN-RANK and ST-RANK, that was reaching a global optimum, while for AP, convergence could be also local. Still, both topological approaches outperform non-toplogical ones by an order of magnitude, as can be seen in Table I.

	VN	**ST**	AP	EX
mean	**2.12**	**1.52**	41.92	50
min	**1**	**1**	24	1
max	**12**	**5**	60	100

TABLE I: Number of objective function evaluations until convergence.

In the second experiment, a limited number of samples is evaluated to demonstrate the concept and show the difference between our BSP solution based on the VN signature and the optimal solution, considering one of the cases when RE was relatively large according to the error distribution shown in Figure 4. Even for small number of samples, in this case only *one* sample was considered, i.e. $N_s = 1$, the topological solution is still very close to the optimal solution (see Figure 5). From Figure 5 it can be also seen that in the case when optimal solution is found, one distinctive topological class can be identified corresponding to the most complex graph, while in the case it was not found, many similar topologies exist and then other factors, e.g. linearization point and noise levels, determine the BSP solution.

V. Conclusions

In this preliminary work, we introduced a novel concept, topological belief space planning to tackle computational complexity aspects of belief space planning (BSP) in high dimensional state spaces. We demonstrated how this general concept can be applied in multi-robot decentralized BSP to overcome main drawbacks of the state-of-the art exhaustive search and announced paths approaches. Evaluating an objective function requires multi-robot belief propagation and, in high dimensional state spaces like in active SLAM problem, constitutes the main computational burden. In topological space, due its lower dimension compared to an embedded state space, computation of certain topological metrics can be performed more efficiently. We demonstrated that topological properties of the underlying factor graph representations of future posterior beliefs exhibit high correlation to information-theoretic cost and proposed a topological metric dependent only on the graph node degrees, the approximation of the Von Neumann entropy, to direct the search for an optimal BSP solution by sub-sampling the topological space. Therefore, our algorithm can be seen as an anytime algorithm that will eventually converge to the optimal solution. Moreover, in cases where a strong correlation exists between the objective and the topological metric, this convergence is by an order of magnitude faster than existing state-of-the-art approaches.

In future work, we plan to investigate incremental aspects of topological BSP and design topological metrics that would be relevant for solving other interesting BSP objectives, e.g. maximizing exploration gain. We also plan to analyze time complexity of different BSP approaches together with presenting formal guarantees of global optimality of our approach in terms of deriving approximation error bounds of the proposed topological metric.

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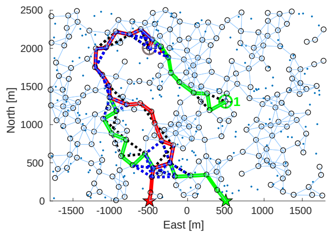
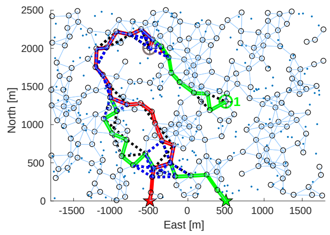
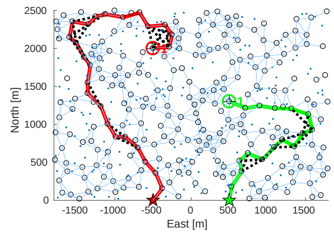


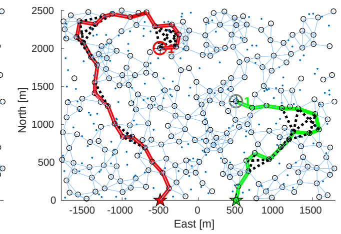
(a) S1 chosen paths



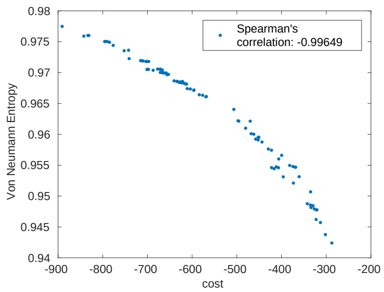
(b) S1 optimal paths



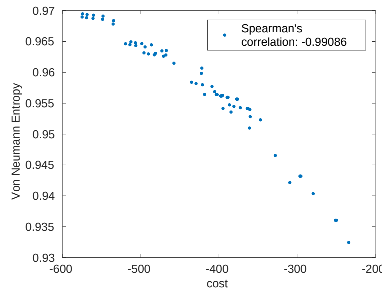
(c) S2 chosen paths



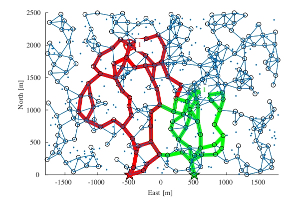
(d) S2 optimal paths



(e) S1 H_{VN} versus cost (3).



(f) S2 H_{VN} versus cost (3).



(g) Candidate paths of two robots (red and green) generated on top of PRM in a single planning session. Starting positions are shown with stars and the goal positions with circles. Blue dots in the map denote environment landmarks.

Fig. 5: Examples from two BSP planning sessions S1 and S2 in which after only *one* sample: **(a)**-**(b)** an optimal solution was found; **(c)**-**(d)** although optimal solution was not found, the chosen solution is relatively close to the optimal (relative error $RE = 2.5\%$). Robot's paths are marked with solid red and green line, multi-robot constraints with dashed blue line, and loop closure constraints with dashed black lines. **(e)** and **(f)** show graph signature's H_{VN} relation to the cost, and demonstrate strong correlation between H_{VN} and the cost (3). Correlation is somewhat smaller in the case when suboptimal solution is found by topological BSP. **(g)** candidate paths per planning session.

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