Computationally Efficient Decision Making Under Uncertainty in High-Dimensional State Spaces

Dmitry Kopitkov and Vadim Indelman

International Conference on Intelligent Robots and Systems (IROS), October 2016
Introduction

- Decision making under uncertainty - fundamental problem in autonomous systems and artificial intelligence

- Examples
  - **Belief space planning** in uncertain/unknown environments (e.g. for autonomous navigation)
  - **Active simultaneous localization and mapping (SLAM)**
  - Informative planning, active sensing
  - Sensor selection, sensor deployment
  - **Multi-agent** informative planning and active SLAM
  - **Graph sparsification** for long-term autonomy
Introduction

- Information-theoretic decision making
  - **Objective**: find action that minimizes an information-theoretic metric (e.g. entropy, information gain, mutual information)
  - **Problem types**: **Unfocused** (entropy of all variables), and **Focused** (entropy only of subset of variables)

- Decision making over **high-dimensional** state spaces is expensive!

\[
X \in \mathbb{R}^n \quad \Lambda \equiv \Sigma^{-1} \in \mathbb{R}^{n \times n}
\]

- Evaluating action impact typically involves determinant calculation: \( O(n^3) \) (smaller for sparse matrices)

- Existing approaches typically calculate posterior information (covariance) matrix for each candidate action, and then its determinant
Key idea

- Resort to **matrix determinant lemma and calculation re-use** techniques for **information-theoretic decision making**

\[
|\Lambda_k + A^T A| = |\Lambda_k| \cdot |I_m + A \cdot \Sigma_k \cdot A^T| \quad \Sigma_k \equiv \Lambda_k^{-1}
\]

- **High-level overview:**
  - **Avoid** calculating determinants of large matrices
  - **Re-use** of calculations
  - Per-action evaluation **does not depend on state** dimension
  - Yet - **exact** and **general** solution
Problem Formulation

- Given action \( a = u_{k:k+L-1} \) and new observations \( Z_{k+1:k+L} \), future belief is:

\[
b[X_{k+L}] = p(X_{k+L}|Z_{0:k+L}, u_{0:k+L-1}) \propto p(X_k|Z_{0:k}, u_{0:k-1}) \prod_{l=k+1}^{k+L} p(x_l|x_{l-1}, u_{l-1}) p(Z_l|X_l^0)
\]

- Planning time

- Motion model

- Measurement likelihood
**Problem Formulation**

\[
b[X_{k+L}] = p(X_{k+L}|Z_{0:k+L}, u_{0:k+L-1}) \propto p(X_k|Z_{0:k}, u_{0:k-1}) \prod_{l=k+1}^{k+L} p(x_l|x_{l-1}, u_{l-1}) p(Z_l|X_l^o)
\]

- Information-theoretic cost (entropy): \( J_{H}(a) = H(p(X_{k+L}|Z_{0:k+L}, u_{0:k+L-1})) \)

- Decision making: \( a^* = \arg \min_{a \in A} J_{H}(a) \quad A \doteq \{ a_1, a_2, \ldots, a_N \} \)

- Gaussian distributions:

\[
\Lambda_{k+L} = \Lambda_k + A^T A
\]

\( J_{H(a)} = \frac{n}{2} \left( 1 + \ln \left( 2\pi \right) \right) - \frac{1}{2} \ln |\Lambda_{k+L}| \)

- Impact evaluation for a candidate action is in the general case: \( O(n^3) \)
Information Gain (IG) & Matrix Determinant Lemma

- Use IG instead of entropy:

\[ J_{IG} (a) = \mathcal{H} (b [X_k]) - \mathcal{H} (b [X_{k+L}]) = \frac{1}{2} \ln \frac{|\Lambda_k + A^T A|}{|\Lambda_k|} \]

- Applying matrix determinant lemma:

\[ |\Lambda_k + A^T A| = |\Lambda_k| \cdot |I_m + A \cdot \Sigma_k \cdot A^T| \]

\[ J_{IG} (a) = \frac{1}{2} \ln |I_m + A \cdot \Sigma_k \cdot A^T| \]

Cheap, given \( \Sigma_k \)!

\[ \Sigma_k \equiv \Lambda_k^{-1} \in \mathbb{R}^{n \times n} \]
\[ A \in \mathbb{R}^{m \times n} \]

Typically: \( m \ll n \)

Examples: sensor deployment, active SLAM
Sparse Structure of Action Jacobian

- Jacobian $A \in \mathbb{R}^{m \times n}$ of action $a$ represents new terms in the joint pdf:

$$b [X_{k+L}] = p(X_{k+L}|Z_{0:k+L}, u_{0:k+L-1}) \propto p(X_k|Z_{0:k}, u_{0:k-1}) \prod_{l=k+1}^{k+L} p(x_l|x_{l-1}, u_{l-1}) p(Z_l|X_l^o)$$

- **Illustration** example:

- Columns of $A$ that are not involved in new terms, contain only zeros
Using Sparsity of Action Jacobian

\[ J_{IG}(a) = \frac{1}{2} \ln |I_m + A \cdot \Sigma_k \cdot A^T| \]

\[ J_{IG}(a) = \frac{1}{2} \ln |I_m + A_C \cdot \Sigma_{k,C} \cdot A_C^T| \]

- \( C \) is set of variables involved in \( A \)
- \( A_C \) is constructed from \( A \) by removing all zero columns
- \( \Sigma_{k,C} \) is prior marginal covariance of \( C \)

- Only few entries from the prior covariance are actually required!
Re-use of Calculation

- **Key observations:**
  - Given $\Sigma_{k}^{M,C}$, calculation of action impact depends only on action Jacobian:
    
    $$J_{IG}(a) = \frac{1}{2} \ln \left| I_{m} + A_{C} \cdot \Sigma_{k}^{M,C} \cdot A_{C}^{T} \right|$$
  - Different candidate actions often share many involved variables

- **Re-use calculations:**
  - Combine variables involved in all candidate actions into set $C_{All}$
  - Perform one-time calculation of $\Sigma_{k}^{M,CAll}$
    - Can be calculated efficiently from square root information matrix
    - Depends on state dimension $n$
  - Calculate $J_{IG}(a)$ for each action, using $\Sigma_{k}^{M,CAll}$
Focused Decision Making

- Posterior entropy over focused variables $X^F_{k+L} \subseteq X_{k+L}$

$$J^F_H(a) = \mathcal{H}(X^F_{k+L}) = \frac{n^F}{2} (1 + \ln(2\pi)) + \frac{1}{2} \ln \left| \Sigma_{k+L}^{M,F} \right|$$

- Applying IG and matrix determinant lemma of Schur complement:

$$|\Lambda_k| = |\Lambda_{k}^{M,F}| \cdot |\Lambda_{k}^{R}|$$

$$J^F_{IG}(a) = \frac{1}{2} \ln \frac{|I_m + A \cdot \Sigma_k \cdot A^T|}{|I_m + A_R \cdot \Sigma_k^R \cdot A_R^T|}$$

$A_R$ – columns of $A$ related to $X^R_k$

$\Sigma_k^R | F$ – is inverse of $\Lambda_{k}^{R}$
Application to Sensor Deployment Problems

- Significant time reduction in *Unfocused* case
Application to Sensor Deployment Problems

- Significant time reduction in *Focused* case
Application to Measurement Selection (in SLAM Context)

‘Standard’: for each action, calculate posterior sqrt information matrix via iSAM2, then its determinant.

D. Kopitkov, V. Indelman, Computationally Efficient Decision Making Under Uncertainty in High-Dimensional State Spaces
Conclusions

- Decision making via matrix determinant lemma and calculation re-use
  - Exact (no approximations applied)
  - General (any measurement model)
  - Per-candidate complexity does not depend on $n$
  - Unfocused and Focused problem formulations
  - Applicable to Sensor Deployment, Measurement Selection, Graph Sparsification, and many more..

- Multiple extensions to be investigated

D. Kopitkov, V. Indelman, Computationally Efficient Decision Making Under Uncertainty in High-Dimensional State Spaces