Nonmyopic Distilled Data Association Belief Space Planning Under Budget Constraints
Supplementary Material

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This document provides supplementary material to the paper [1]. Therefore, it should not be considered a self-contained document, but instead regarded as an appendix of [1]. Throughout this report, all notations and definitions are with compliance to the ones presented in [1].

A Notations

Given a simplified belief b^s_{k+n} with M^s_{k+n} components and a belief component $p_{k+n} \notin M^s_{k+n}$ with associated weight w^p_{k+n}, we denote $M^{s+1}_{k+n} \triangleq M^s_{k+n} \cup p_{k+n}$. We also further denote the bounds in Theorem 2 as $\mathcal{LB}\left[\mathcal{H}_{k+n}|M^s_{k+n}\right]$,$\mathcal{UB}\left[\mathcal{H}_{k+n}|M^s_{k+n}\right]$, and the bounds in Theorem 3 as $\mathcal{LB}\left[\eta_{k+n}|M^s_{k+n}\right]$,$\mathcal{UB}\left[\eta_{k+n}|M^s_{k+n}\right]$, i.e. with respect to M^s_{k+n} components of the simplified belief b^s_{k+n}.

B Proofs

B.1 Theorem 1

Given M_{k+n} belief components, we split the cost function (eq. (20) in [1]) based on components in and outside M^s_{k+n}

$$\mathcal{H}_{k+n} = -\sum_{r \in M^s_{k+n}} \frac{w^r_{k+n}}{\eta_{k+n}} log\left(\frac{w^r_{k+n}}{\eta_{k+n}}\right) - \sum_{r \in \neg M^s_{k+n}} \frac{w^r_{k+n}}{\eta_{k+n}} log\left(\frac{w^r_{k+n}}{\eta_{k+n}}\right). \tag{1}$$

Using basic log properties and $w^{s,r}_{k+n} \triangleq \frac{w^r_{k+n}}{w^{m,s}_{k+n}}$ where $w^{m,s}_{k+n} \triangleq \sum_{m \in M^s_{k+n}} w^m_{k+n}$ completes the proof

$$\begin{aligned}
\mathcal{H}_{k+n} &= -\sum_{r \in M^s_{k+n}} \frac{w^{m,s}_{k+n} w^{s,r}_{k+n}}{\eta_{k+n}} log\left(w^{m,s}_{k+n} w^{s,r}_{k+n}\right) + \sum_{r \in M^s_{k+n}} \frac{w^{m,s}_{k+n} w^{s,r}_{k+n}}{\eta_{k+n}} log\left(\eta_{k+n}\right) - \sum_{r \in \neg M^s_{k+n}} \frac{w^r_{k+n}}{\eta_{k+n}} log\left(\frac{w^r_{k+n}}{\eta_{k+n}}\right) \\
&= \frac{w^{m,s}_{k+n}}{\eta_{k+n}}\left[-\sum_{r \in M^s_{k+n}} w^{s,r}_{k+n} log\left(w^{m,s}_{k+n} w^{s,r}_{k+n}\right) + \sum_{r \in M^s_{k+n}} w^{s,r}_{k+n} log\left(\eta_{k+n}\right)\right] - \sum_{r \in \neg M^s_{k+n}} \frac{w^r_{k+n}}{\eta_{k+n}} log\left(\frac{w^r_{k+n}}{\eta_{k+n}}\right) \\
&= \frac{w^{m,s}_{k+n}}{\eta_{k+n}}\left[-\sum_{r \in M^s_{k+n}} w^{s,r}_{k+n} log\left(w^{s,r}_{k+n}\right) - log\left(w^{m,s}_{k+n}\right) + log\left(\eta_{k+n}\right)\right] - \sum_{r \in \neg M^s_{k+n}} \frac{w^r_{k+n}}{\eta_{k+n}} log\left(\frac{w^r_{k+n}}{\eta_{k+n}}\right) \\
&= \frac{w^{m,s}_{k+n}}{\eta_{k+n}}\left[\mathcal{H}^s_{k+n} + log\left(\frac{\eta_{k+n}}{w^{m,s}_{k+n}}\right)\right] - \sum_{r \in \neg M^s_{k+n}} \frac{w^r_{k+n}}{\eta_{k+n}} log\left(\frac{w^r_{k+n}}{\eta_{k+n}}\right).
\end{aligned} \tag{2}$$

■

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B.2 Theorem 2

The last term in (2) is non negative as all posterior weights are at most 1 by definition. Thus, removing this term and using the bounds over η_{k+n} in Theorem 3 we immediately get the lower bound

$$\mathcal{LB}\left[\mathcal{H}_{k+n}\right] = \frac{w^{m,s}_{k+n}}{\mathcal{UB}\left[\eta_{k+n}\right]}\left[\mathcal{H}^{s}_{k+n} + log\left(\frac{\mathcal{LB}\left[\eta_{k+n}\right]}{w^{m,s}_{k+n}}\right)\right]. \quad (3)$$

For the upper bound, we first define

$$\gamma \triangleq \sum_{r\in\neg M^{s}_{k+n}} \frac{w^{r}_{k+n}}{\eta_{k+n}} = 1 - \sum_{r\in M^{s}_{k+n}} \frac{w^{r}_{k+n}}{\eta_{k+n}}. \quad (4)$$

Using the log sum inequality [2]

$$\sum_{i}^{n} a_i \cdot log\left(\frac{a_i}{b_i}\right) \geq a \cdot log\left(\frac{a}{b}\right) \text{ where} \sum_{i}^{n} a_i = a, \sum_{i}^{n} b_i = b \quad (5)$$

with $a_i = \frac{w^{r}_{k+n}}{\eta_{k+n}}$ and $b_i = 1$, we bound the last term in (2)

$$\sum_{r\in\neg M^{s}_{k+n}} \frac{w^{r}_{k+n}}{\eta_{k+n}} log\left(\frac{w^{r}_{k+n}}{\eta_{k+n}}\right) \geq \gamma log\left(\frac{\gamma}{\left|\neg M^{s}_{k+n}\right|}\right). \quad (6)$$

Substituting (6) into (2); using the bounds over η_{k+n} from Theorem 3; and since by definition $0 \leq \gamma \leq 1$, we get the upper bound

$$\mathcal{UB}\left[\mathcal{H}_{k+n}\right] = \frac{w^{m,s}_{k+n}}{\mathcal{LB}\left[\eta_{k+n}\right]}\left[\mathcal{H}^{s}_{k+n} + log\left(\frac{\mathcal{UB}\left[\eta_{k+n}\right]}{w^{m,s}_{k+n}}\right)\right] - \bar{\gamma} log\left(\frac{\bar{\gamma}}{\left|\neg M^{s}_{k+n}\right|}\right), \quad (7)$$

where $\bar{\gamma} = 1 - \sum_{r\in M^{s}_{k+n}} \frac{w^{r}_{k+n}}{\mathcal{UB}[\eta_{k+n}]}$ and $\left|\neg M^{s}_{k+n}\right| > 2$. ■

B.3 Corollary 1

Given that $M^{s}_{k+n} = M_{k+n}$ it holds by definition that $\eta_{k+n} = w^{m,s}_{k+n}$ and $\mathcal{H}_{k+n} = \mathcal{H}^{s}_{k+n}$. Substituting back into (3) and using Corollary 2 we get

$$\lim_{M^{s}_{k+n}\to M_{k+n}} \mathcal{LB}\left[\mathcal{H}_{k+n}\right] = \frac{w^{m,s}_{k+n}}{\eta_{k+n}}\left[\mathcal{H}_{k+n} + log\left(\frac{\eta_{k+n}}{w^{m,s}_{k+n}}\right)\right] = \mathcal{H}_{k+n}. \quad (8)$$

It is also straightforward that $M^{s}_{k+n} = M_{k+n} \Rightarrow \bar{\gamma} = 0$. As such, similarly to the lower bound, it immediately holds that $\mathcal{H}_{k+n} = \lim_{M^{s}_{k+n}\to M_{k+n}} \mathcal{UB}\left[\mathcal{H}_{k+n}\right]$. ■

B.4 Theorem 3

By definition $\eta_{k+n} = \sum_{r\in M_{k+n}} w^{r}_{k+n}$. Splitting this sum, we rewrite η_{k+n} as

$$\eta_{k+n} = \sum_{r\in M^{s}_{k+n}} w^{r}_{k+n} + \sum_{r\in\neg M^{s}_{k+n}} w^{r}_{k+n}. \quad (9)$$

The second term in (9) is positive by definition. As such, removing it we immediately get the lower bound

$$\mathcal{LB}\left[\eta_{k+n}\right] = \sum_{r\in M^{s}_{k+n}} w^{r}_{k+n}. \quad (10)$$

For the upper bound, we first rewrite the second term in (9) as

$$\sum_{r\in\neg M^{s}_{k+n}} w^{r}_{k+n} = \sum_{r\in\neg M^{s}_{k+n}} \prod_{i=0}^{n} w^{r}_{k+i} = \sum_{r\in\neg M^{s}_{k+n}} w^{r}_{k} \prod_{i=1}^{n} w^{r}_{k+i}. \quad (11)$$

Each w^r_{k+i} is defined as

$$\int_{x_{k+i}} \mathbb{P}(Z_{k+i}|\beta^r_{k+i}, x_{k+i})\mathbb{P}(\beta^r_{k+i}|x_{k+i})\mathbb{P}(x_{k+i}|H^-_{k+i}, \beta^r_{1:k+i}), \tag{12}$$

which can also be bounded as presented in Theorem 4 in [3]. The joint measurement likelihood term is a product of probability distribution functions, all given a priori, and can be bounded using a known maximum value σ^i. The term $\mathbb{P}(\beta^r_{k+i}|x_{k+i})$ represents the probability for the rth data association realization given x_{k+i} and can bounded by 1 representing, for example, an indicator function for landmarks that are within the field of view. Finally, for every hypothesis r it holds that $\int_{x_{k+i}} \mathbb{P}(x_{k+i}|H^-_{k+i}, \beta^r_{1:k+i}) = 1$. By definition, $\sum_{r\in M_k} w^r_k = 1$ and each component at time k generates $\frac{|M_{k+n}|}{|M_k|}$ at time $k+n$, thus

$$\sum_{r\in M_{k+n}} w^r_k = \sum_{r\in M^s_{k+n}} w^r_k + \sum_{r\in \neg M^s_{k+n}} w^r_k = \frac{|M_{k+n}|}{|M_k|}. \tag{13}$$

As such, we can bound (11) as

$$\sum_{r\in \neg M^s_{k+n}} w^r_{k+n} \leq \sum_{r\in \neg M^s_{k+n}} w^r_k \prod_{i=1}^{n} \sigma^i = \left(\frac{|M_{k+n}|}{|M_k|} - \sum_{r\in M^s_{k+n}} w^r_k \right) \prod_{i=1}^{n} \sigma^i. \tag{14}$$

Substituting back into (9) we get the upper bound

$$\mathcal{UB}\left[\eta_{k+n}\right] = \sum_{r\in M^s_{k+n}} w^r_{k+n} + \left(\frac{|M_{k+n}|}{|M_k|} - \sum_{r\in M^s_{k+n}} w^r_k \right) \prod_{i=1}^{n} \sigma^i. \tag{15}$$

■

B.5 Corollary 2

Given that $M^s_{k+n} = M_{k+n}$ it holds by definition that $\eta_{k+n} = w^{m,s}_{k+n}$. Substituting back into (10) we get

$$\lim_{M^s_{k+n}\to M_{k+n}} \mathcal{LB}\left[\eta_{k+n}\right] = \sum_{r\in M^s_{k+n}} w^r_{k+n} = w^{m,s}_{k+n} = \eta_{k+n}. \tag{16}$$

It is also straightforward that $M^s_{k+n} = M_{k+n} \Rightarrow \frac{|M_{k+n}|}{|M_k|} - \sum_{r\in M^s_{k+n}} w^r_k = 0$ As such, similarly to the lower bound, it immediately holds that $\eta_{k+n} = \lim_{M^s_{k+n}\to M_{k+n}} \mathcal{UB}\left[\eta_{k+n}\right]$. ■

C Recursive update rules

C.1 Incrementally adapting $\mathcal{LB}\left[\eta_{k+n}\right], \mathcal{LB}\left[\eta_{k+n}\right]$

Given a belief component $p_{k+n} \notin M^s_{k+n}$ with associated weight w^p_{k+n}, we first derive a recursive update rule for the lower bound

$$\mathcal{LB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right] = \sum_{r\in M^{s+1}_{k+n}} w^r_{k+n} = w^p_{k+n} + \sum_{r\in M^s_{k+n}} w^r_{k+n} = w^p_{k+n} + \mathcal{LB}\left[\eta_{k+n}|M^s_{k+n}\right]. \tag{17}$$

The recursive update rule for the upper bound is given by

$$\begin{aligned}
\mathcal{UB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right] &= \sum_{r\in M^{s+1}_{k+n}} w^r_{k+n} + \left(\frac{|M_{k+n}|}{|M_k|} - \sum_{r\in M^{s+1}_{k+n}} w^r_k \right) \prod_{i=1}^{n} \sigma^i \\
&= w^p_{k+n} + \sum_{r\in M^s_{k+n}} w^r_{k+n} + \left(\frac{|M_{k+n}|}{|M_k|} - w^p_k - \sum_{r\in M^s_{k+n}} w^r_k \right) \prod_{i=1}^{n} \sigma^i \\
&= w^p_{k+n} - w^p_k \prod_{i=1}^{n} \sigma^i + \mathcal{UB}\left[\eta_{k+n}|M^s_{k+n}\right],
\end{aligned} \tag{18}$$

where $\sigma^i \triangleq \max\left(\mathbb{P}\left(Z_{k+i}|x_{k+i}\right)\right)$ and w^p_k is the prior weight at time k of the belief component p_{k+n}. ■

C.2 Incrementally adapting $\mathcal{LB}\left[\mathcal{H}_{k+n}\right], \mathcal{LB}\left[\mathcal{H}_{k+n}\right]$

Deriving a direct recursive update rule for these bounds is not trivial. Instead, we show how each term in $\mathcal{LB}\left[\mathcal{H}_{k+n}|M^{s+1}_{k+n}\right]$, $\mathcal{UB}\left[\mathcal{H}_{k+n}|M^{s+1}_{k+n}\right]$ can be incrementally updated individually.

Given a belief component $p_{k+n} \notin M^s_{k+n}$ with associated weight w^p_{k+n}, we first derive a recursive update rule for the cost over the simplified belief b^{s+1}_{k+n}, i.e. containing $M^{s+1}_{k+n} \triangleq M^s_{k+n} \cup p_{k+n}$ components

$$
\begin{aligned}
\mathcal{H}^{s+1}_{k+n} \triangleq c\left(b^{s+1}_{k+n}\right) &= -\sum_{r\in M^{s+1}_{k+n}} \frac{w^r_{k+n}}{\sum_{r\in M^{s+1}_{k+n}} w^r_{k+n}} log\left(\frac{w^r_{k+n}}{\sum_{r\in M^{s+1}_{k+n}} w^r_{k+n}}\right) \\
&= -\frac{w^p_{k+n}}{\sum_{r\in M^{s+1}_{k+n}} w^r_{k+n}} log\left(\frac{w^p_{k+n}}{\sum_{r\in M^{s+1}_{k+n}} w^r_{k+n}}\right) - \sum_{r\in M^s_{k+n}} \frac{w^r_{k+n}}{\sum_{r\in M^{s+1}_{k+n}} w^r_{k+n}} log\left(\frac{w^r_{k+n}}{\sum_{r\in M^{s+1}_{k+n}} w^r_{k+n}}\right) \\
&= -\frac{w^p_{k+n}}{w^p_{k+n} + \sum_{r\in M^s_{k+n}}} log\left(\frac{w^p_{k+n}}{w^p_{k+n} + \sum_{r\in M^s_{k+n}}}\right) - \sum_{r\in M^s_{k+n}} \frac{w^r_{k+n}}{w^p_{k+n} + \sum_{r\in M^s_{k+n}} w^r_{k+n}} log\left(\frac{w^r_{k+n}}{w^p_{k+n} + \sum_{r\in M^s_{k+n}} w^r_{k+n}}\right) \\
&= -\frac{w^p_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}} log\left(\frac{w^p_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}}\right) - \sum_{r\in M^s_{k+n}} \frac{w^r_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}} log\left(\frac{w^r_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}}\right) \\
&= -\frac{w^p_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}} log\left(\frac{w^p_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}}\right) - \frac{w^{m,s}_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}} \sum_{r\in M^s_{k+n}} \frac{w^r_{k+n}}{w^{m,s}_{k+n}} \left[log\left(\frac{w^r_{k+n}}{w^{m,s}_{k+n}}\right) + log\left(\frac{w^{m,s}_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}}\right)\right] \\
&= -\frac{w^p_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}} log\left(\frac{w^p_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}}\right) - \frac{w^{m,s}_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}} \left[-\mathcal{H}^s_{k+n} + \sum_{r\in M^s_{k+n}} \frac{w^r_{k+n}}{w^{m,s}_{k+n}} log\left(\frac{w^{m,s}_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}}\right)\right] \\
&= -\frac{w^p_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}} log\left(\frac{w^p_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}}\right) - \frac{w^{m,s}_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}} \left[-\mathcal{H}^s_{k+n} + log\left(\frac{w^{m,s}_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}}\right) \sum_{r\in M^s_{k+n}} \frac{w^r_{k+n}}{w^{m,s}_{k+n}}\right] \\
&= -\frac{w^p_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}} log\left(\frac{w^p_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}}\right) - \frac{w^{m,s}_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}} \left[log\left(\frac{w^{m,s}_{k+n}}{w^p_{k+n} + w^{m,s}_{k+n}}\right) - \mathcal{H}^s_{k+n}\right].
\end{aligned}
\tag{19}
$$

Using (19) and the recursive update rules derived in Section C.1, we get a recursive update rule for the lower bound

$$
\begin{aligned}
\mathcal{LB}\left[\mathcal{H}_{k+n}|M^{s+1}_{k+n}\right] &= \frac{\sum_{r\in M^{s+1}_{k+n}} w^r_{k+n}}{\mathcal{UB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right]} \left[\mathcal{H}^{s+1}_{k+n} + log\left(\frac{\mathcal{LB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right]}{\sum_{r\in M^{s+1}_{k+n}} w^r_{k+n}}\right)\right] \\
&= \frac{w^p_{k+n} + w^{m,s}_{k+n}}{\mathcal{UB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right]} \left[\mathcal{H}^{s+1}_{k+n} + log\left(\frac{\mathcal{LB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right]}{w^p_{k+n} + w^{m,s}_{k+n}}\right)\right].
\end{aligned}
\tag{20}
$$

Similarly, we also get a recursive update rule for the upper bound

$$
\begin{aligned}
\mathcal{UB}\left[\mathcal{H}_{k+n}|M^{s+1}_{k+n}\right] &= \frac{\sum_{r\in M^{s+1}_{k+n}} w^r_{k+n}}{\mathcal{LB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right]} \left[\mathcal{H}^{s+1}_{k+n} + log\left(\frac{\mathcal{UB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right]}{\sum_{r\in M^{s+1}_{k+n}} w^r_{k+n}}\right)\right] - \bar{\gamma} log\left(\frac{\bar{\gamma}}{|\neg M^{s+1}_{k+n}|}\right) \\
&= \frac{w^p_{k+n} + w^{m,s}_{k+n}}{\mathcal{LB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right]} \left[\mathcal{H}^{s+1}_{k+n} + log\left(\frac{\mathcal{UB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right]}{w^p_{k+n} + w^{m,s}_{k+n}}\right)\right] - \bar{\gamma} log\left(\frac{\bar{\gamma}}{|\neg M^{s+1}_{k+n}|}\right),
\end{aligned}
\tag{21}
$$

where $\bar{\gamma} = 1 - \sum_{r\in M^{s+1}_{k+n}} \frac{w^r_{k+n}}{\mathcal{UB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right]} = 1 - \frac{w^p_{k+n} + w^{m,s}_{k+n}}{\mathcal{UB}\left[\eta_{k+n}|M^{s+1}_{k+n}\right]}$ and $|\neg M^{s+1}_{k+n}| > 2$. ■

References

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