Qualitative Belief Space Planning via Compositions

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MSc Seminar, January 23, 2022

ANPL Autonomous Navigation and Perception Lab

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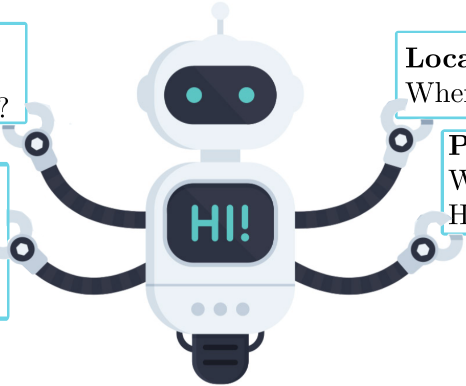
Introduction - Autonomous Systems

Introduction - common problems in robotics

Mapping
What's around me?

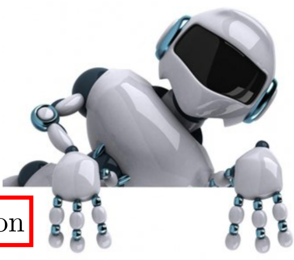
SLAM
Simultaneous Localization and Mapping



Localization
Where am I?

Planning
Where to go?
How to get there?

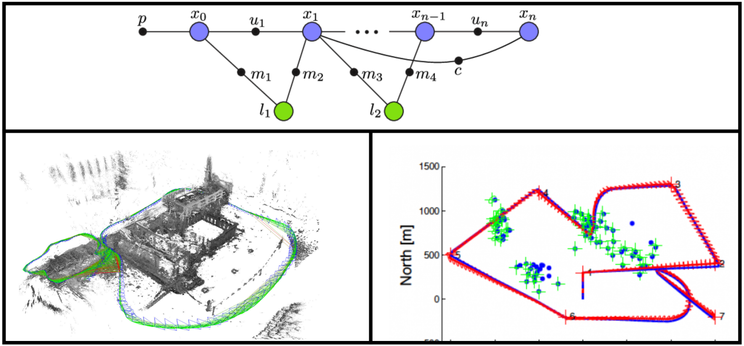
Agenda



- Introduction and Motivation
- Qualitative Belief Space Planning via Compositions
- Compositions Calculi
- Summary and Conclusions

Metric Approaches

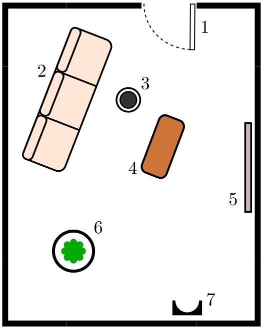
- Classical robotics applications rely on accurate metric estimations of the environment and robot's location to accomplish their aims.
- Big optimization problems, noise-sensitive



- While maintaining accurate information is often essential, it might be unnecessary in some cases.

Qualitative Approaches - Motivation

- Consider the following living-room scene:

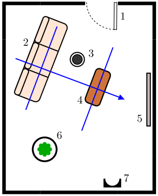


1 - door
2 - sofa
3 - i-robot
4 - cofee table
5 - TV
6 - flowerpot
7 - charger

- Relying on coarse relationships between the different objects may be sufficient to maneuver within the room successfully
- These relationships are known as **Qualitative Spatial Relationships** or **QSR** in short

Qualitative Approaches - Motivation

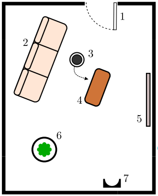
- The map can be described through qualitative relationships between objects (triplets in our case):



Flowerpot relative to sofa-table frame: "**Middle-Right**"

1 - door
2 - sofa
3 - i-robot
4 - cofee table
5 - TV
6 - flowerpot
7 - charger

- Qualitative localization might be good enough in some cases:

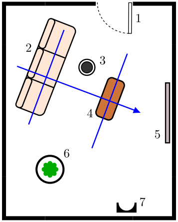


To clean under the table, pass safely between its legs

Qualitative Approaches - Motivation

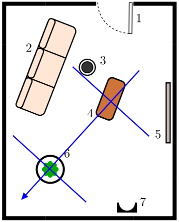
1 - door
2 - sofa
3 - i-robot
4 - cofee table
5 - TV
6 - flowerpot
7 - charger

Example 3:



Flowerpot relative to sofa-table frame: "**Middle-Right**"

+



charger relative to table-flowerpot frame: "**Middle-Left**"

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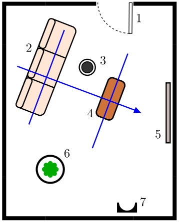
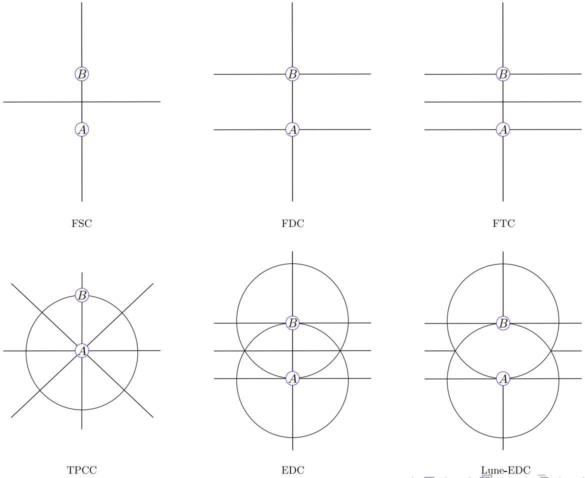


charger relative to sofa-table frame: "**Top-Right**"

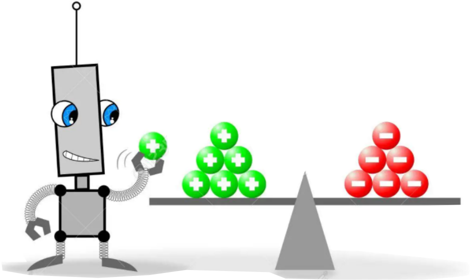
- Given two source triplets, we can conclude a third one under some conditions. This operation is known as **Composition**.

- There are several partitioning types in the literature.
- In general, each triplet can be defined based on a different partitioning.

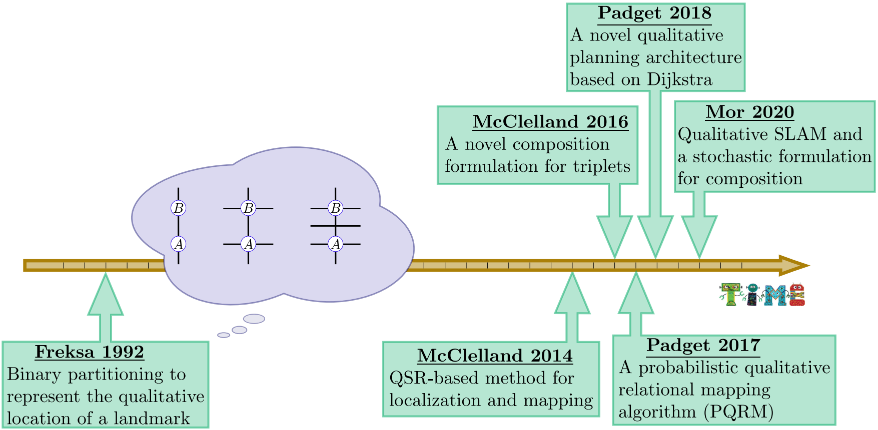


Qualitative Approach - pros and cons

- Pros:
  - Noise robustness - suitable for low-cost platforms
  - Breaking the original problem into small ones
  - Sparse map representations
  - Sometimes it's good enough
- Cons:
  - Less accurate
  - Limited to specific tasks



Related Work



Related Work

Paper	Localization	Mapping	Planning
Levit1990	✓		
Freksa1992a		✓	
Freksa1992b		✓	
Schliender1993	✓		
Schlieder1995		✓	
Wagner2004	✓		
Moratz2011		✓	
Mossakowsky2012		✓	
McClelland2014	✓	✓	
McClelland2016	✓	✓	
Padget2017		✓	
Padget2018			✓
Mor2020	✓	✓	

Agenda

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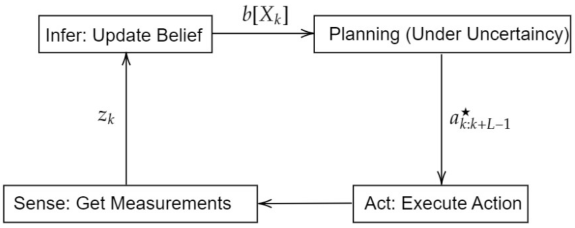
Qualitative BSP - Contributions

- A first-of-its-kind Qualitative Belief Space Planning formulation
- Compositions incorporation to improve results
- A novel cost function that globally measures metric path length

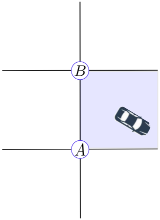
A paper regarding this part is in progress:
"Qualitative BSP via Compositions"

Plan-act-sense-infer framework



- We focus on the planning phase
- We formulate the problem as Belief Space Planning (BSP), considering a qualitative framework

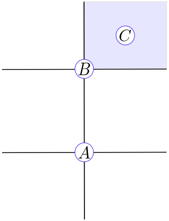
- **Robot's State**



Notations:

Robot's state at time step t, relative to frame F_t	$\mathcal{S}^{F_t:X_t}$
For example...	$\mathcal{S}^{AB:X_t}$
Set of Robot's States between time steps t and t'	$\mathcal{S}^{X_{t:t'}}$

- **Triplet's State**

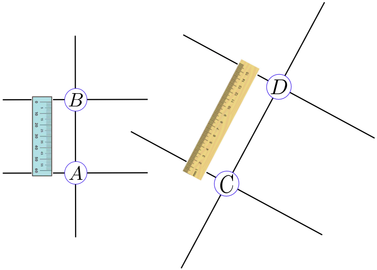


Notations:

State of the triplet τ	$\mathcal{S}^{\tau}$
For example...	$\mathcal{S}^{AB:C}$
Set of available triplets' states at time t (where: $\mathcal{M}_t \triangleq \{\tau_1, \ldots, \tau_{m_t}\}$)	$\mathcal{S}^{\mathcal{M}_t}$

Basic Terms and Notations

- **Frame's Scale**: the global metric distanse between the two landmarks creating the frame.



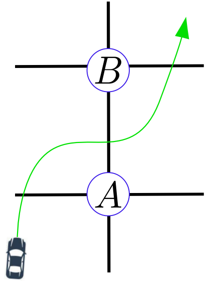
Notations:

Global scale of frame F	$\mathcal{S}^F$
For example...	$\mathcal{S}^{AB}$
Set of available frames' scales at time t (where: $\mathbb{F}_t \triangleq \{F_1, \ldots, F_{p_t}\}$)	$\mathcal{S}^{\mathbb{F}_t}$

- Essential for evaluating future observation's likelihood and metric path's length

Qualitative Action

- Enables the robot to move from one qualitative state to another, considering a specific reference frame.

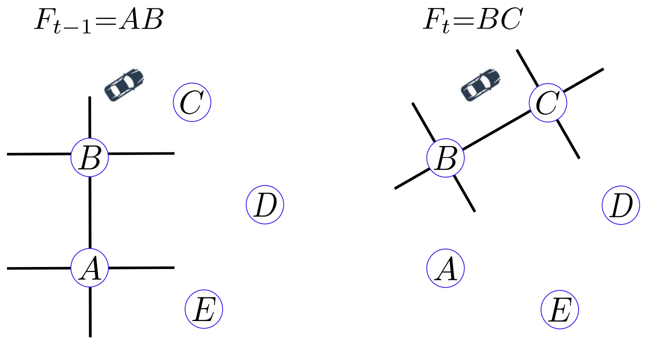


- We assume a probabilistic transition model, given by:

$$\mathbb{P}(\mathcal{S}^{F_t:X_{t+1}}|\mathcal{S}^{F_t:X_t}, a_t^{\mathrm{q}})$$

Link Action

- Allows the robot to switch between different reference frames.



- We assume a probabilistic transition model, given by:

$$\mathbb{P}(\mathcal{S}^{F_t:X_t}|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^{AB:C},a_t^{\text{Link}}), \quad \text{where } a_t^{\text{Link}}=\{AB, BC\}$$

Qualitative Belief Definition

- Consider k as the current time step. The belief defined as a posterior distribution over over the states of the robot, landmark triplets, and frames' scales:

$$\boxed{b_k \triangleq \mathbb{P}(\mathcal{S}^{X_{1:k}}, \mathcal{S}^{\mathcal{M}_k}, \mathcal{S}^{\mathbb{F}_k} | \mathcal{H}_k)}$$

- $\mathcal{H}_k$ denotes the history of applied actions, measurements and data associations:

$$\mathcal{H}_k \triangleq \{a_{1:k}, z_{1:k}, \beta_{1:k}\}, \text{ where } a_t \triangleq \{a_t^q, a_t^{\text{Link}}\}, \forall t \in \{1, \ldots, k\}$$

Qualitative BSP - Problem Statement

- Considering a future horizon of L look-ahead steps, the objective function defined as:

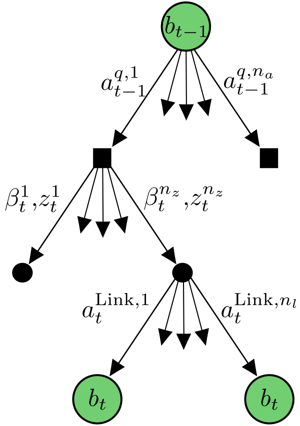
$$J(b_k, a_{k:k+L-1}) \triangleq \mathop{\mathbb{E}}_{z_{k+1:k+L}} \left[\sum_{l=1}^{L-1} c_l(b_{k+l}, a_{k+l-1}) + c_L(b_{k+L}) \right]$$

- We aim to find an optimal sequence of actions, that minimizes the objective:

$$\boxed{a^*_{k:k+L-1} = \mathop{\arg\min}_{a_{k:k+L-1}} J(b_k, a_{k:k+L-1})}$$

Qualitative BSP - Belief Tree

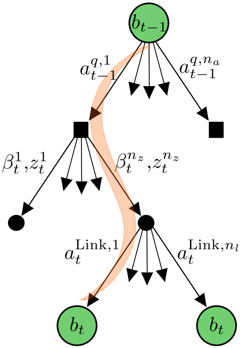
- Planning is done by constructing a belief tree, reflecting the propagated belief considering various possible future developments



Qualitative BSP - Belief Update Step

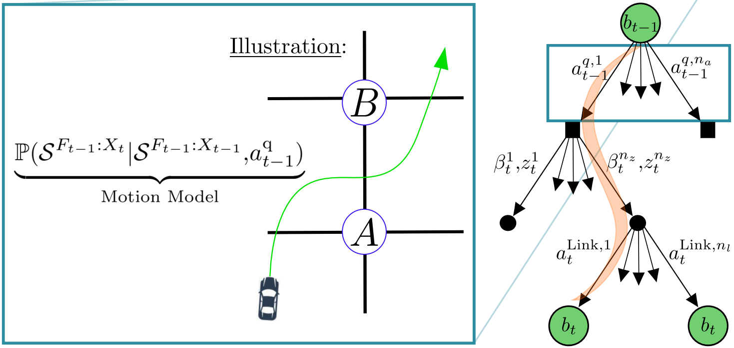
- Given the candidate tuple $a_t^q, \beta_t, z_t, a_t^{Link}$, we update the belief recursively, as follows:

$$
\begin{aligned}
b_t &\triangleq \mathbb{P}(\mathcal{S}^{X_{1:t}}, \mathcal{S}^{\mathcal{M}_t}, \mathcal{S}^{\mathbb{F}_t} | \mathcal{H}_t) \\
&= \sum_{\mathcal{S}^{F_{t-1}:X_t}} \underbrace{\mathbb{P}(\mathcal{S}^{F_t:X_t} | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, a_t^{Link})}_{\text{Link Model}} \eta_t \underbrace{\mathbb{P}(z_t | \beta_t, \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, \mathcal{S}^{F_{t-1}})}_{\text{Measurement Model}} \underbrace{\mathbb{P}(\beta_t | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, \mathcal{S}^{F_{t-1}})}_{\text{Association Model}} \underbrace{\mathbb{P}(\mathcal{S}^{F_{t-1}:X_t} | \mathcal{S}^{F_{t-1}:X_{t-1}}, a_{t-1}^{\mathrm{q}})}_{\text{Motion Model}} b_{t-1}
\end{aligned}
$$



- Qualitative Motion Model:

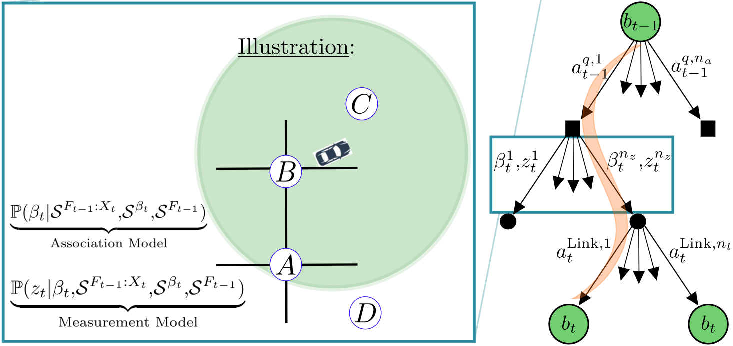
$$b_t \triangleq \mathbb{P}(\mathcal{S}^{X_{1:t}}, \mathcal{S}^{\mathcal{M}_t}, \mathcal{S}^{\mathbb{F}_t} | \mathcal{H}_t)$$
$$= \sum_{\mathcal{S}^{F_{t-1}:X_t}} \underbrace{\mathbb{P}(\mathcal{S}^{F_t:X_t} | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, a_t^{Link})}_{\text{Link Model}} \eta_t \underbrace{\mathbb{P}(z_t | \beta_t, \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, \mathcal{S}^{F_{t-1}})}_{\text{Measurement Model}} \underbrace{\mathbb{P}(\beta_t | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, \mathcal{S}^{F_{t-1}})}_{\text{Association Model}} \boxed{\underbrace{\mathbb{P}(\mathcal{S}^{F_{t-1}:X_t} | \mathcal{S}^{F_{t-1}:X_{t-1}}, a^{\mathrm{q}}_{t-1})}_{\text{Motion Model}}} b_{t-1}$$



- Measurement Model:

$$b_t \triangleq \mathbb{P}(\mathcal{S}^{X_{1:t}}, \mathcal{S}^{\mathcal{M}_t}, \mathcal{S}^{\mathbb{F}_t} | \mathcal{H}_t)$$

$$= \sum_{\mathcal{S}^{F_{t-1}:X_t}} \underbrace{\mathbb{P}(\mathcal{S}^{F_t:X_t} | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, a_t^{Link})}_{\text{Link Model}} \eta_t \boxed{\underbrace{\mathbb{P}(z_t | \beta_t, \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, \mathcal{S}^{F_{t-1}})}_{\text{Measurement Model}} \underbrace{\mathbb{P}(\beta_t | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, \mathcal{S}^{F_{t-1}})}_{\text{Association Model}}} \underbrace{\mathbb{P}(\mathcal{S}^{F_{t-1}:X_t} | \mathcal{S}^{F_{t-1}:X_{t-1}}, a_{t-1}^{q})}_{\text{Motion Model}} b_{t-1}$$



Qualitative BSP - Belief Update Step

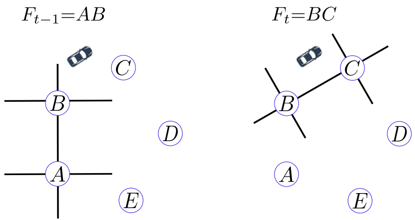
- Link Model:

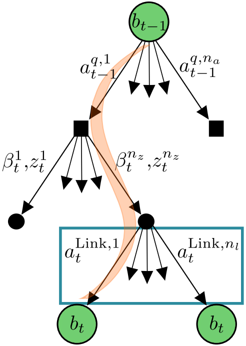
$$b_t \triangleq \mathbb{P}(\mathcal{S}^{X_{1:t}}, \mathcal{S}^{\mathcal{M}_t}, \mathcal{S}^{\mathbb{F}_t} | \mathcal{H}_t)$$

$$= \sum_{\mathcal{S}^{F_{t-1}:X_t}} \boxed{\underbrace{\mathbb{P}(\mathcal{S}^{F_t:X_t} | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, a_t^{Link})}_{\text{Link Model}}} \eta_t \underbrace{\mathbb{P}(z_t | \beta_t, \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, \mathcal{S}^{F_{t-1}})}_{\text{Measurement Model}} \underbrace{\mathbb{P}(\beta_t | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, \mathcal{S}^{F_{t-1}})}_{\text{Association Model}} \underbrace{\mathbb{P}(\mathcal{S}^{F_{t-1}:X_t} | \mathcal{S}^{F_{t-1}:X_{t-1}}, a_{t-1}^{q})}_{\text{Motion Model}} b_{t-1}$$

$$\underbrace{\mathbb{P}(\mathcal{S}^{F_t:X_t} | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, a_t^{Link})}_{\text{Link Model}}$$

Illustration:

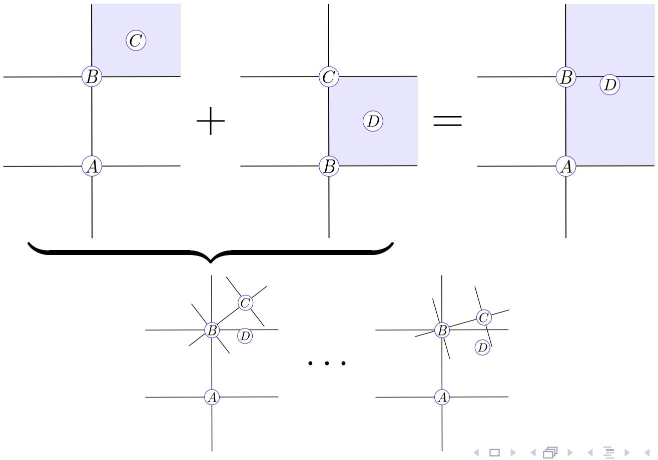




Composition - Spatial Information Propagation

- Given two triplets, we can evaluate the third
- Source triplets must share two landmarks in common

$$\text{Compose}(\text{AB:C, BC:D}) = \text{AB:D}$$



Qualitative BSP - Incorporating Compositions

- Incorporating compositions within our algorithm further improves planning results in two ways:
  - It allows us to deal with a broader range of scenarios, i.e., in some cases, a plan can be found only via compositions
  - We can find better plans, i.e., ones with a lower objective

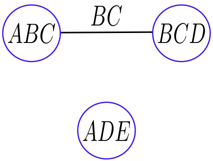
Link-Graph

- A topological representation of the qualitative map
- Triplets are nodes, and frames are edges

Definition 1. A Link-Graph is a graph $G = (V, E)$ where:

1) Each node $v \in V$ represents a triplet of landmarks, i.e., $v = \{\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3\}$.
2) There is an edge $e = (v_1, v_2) \in E$ if and only if $|v_1 \bigcap v_2| = 2$ (i.e., nodes v_1 and v_2 share exactly 2 landmarks in common).

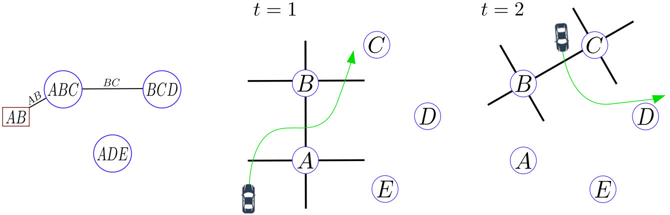
For example:



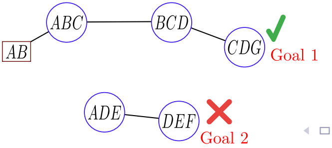
Link-Graph and robot's mobility

- Link-Graph represents mobility between frames:

Lemma 2. A direct *Link* from F_1 to F_2 is feasible based on a triplet τ, if $F_i \subseteq \tau$, $\forall i \in \{1,2\}$, or, in terms of a *Link-Graph*, if the edges representing F_1 and F_2 are connected to the node representing τ.

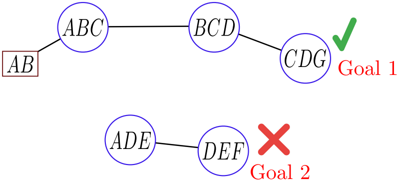


- Conclusion: A Link-Graph's path encodes a feasible sequence of link actions

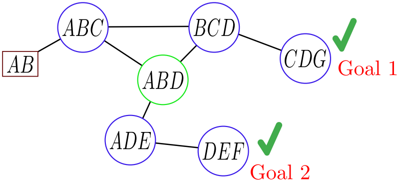


Link-Graph and Compositions

- Conclusion: A Link-Graph's path encodes a feasible sequence of link actions



- Using compositions, we can augment our Link-Graph and improve connectivity:



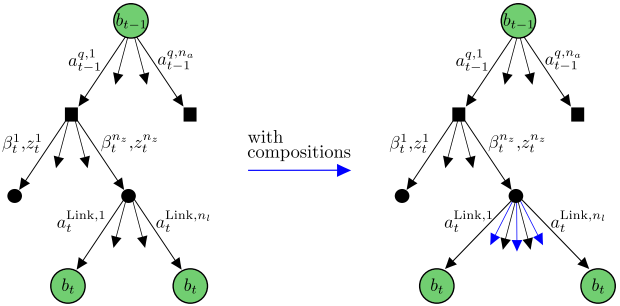
- Consequently, in some scenarios, a valid plan can be found exclusively using compositions

Qualitative BSP via Compositions

$$b_t \triangleq \mathbb{P}(\mathcal{S}^{X_{1:t}}, \mathcal{S}^{\mathcal{M}_t}, \mathcal{S}^{\mathbb{F}_t} | \mathcal{H}_t)$$

$$= \sum_{\mathcal{S}^{F_{t-1}:X_t}} \underbrace{\mathbb{P}(\mathcal{S}^{F_t:X_t} | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, a_t^{Link})}_{\text{Link Model}} \eta_t \underbrace{\mathbb{P}(z_t | \beta_t, \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, \mathcal{S}^{F_{t-1}})}_{\text{Measurement Model}} \underbrace{\mathbb{P}(\beta_t | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{\beta_t}, \mathcal{S}^{F_{t-1}})}_{\text{Association Model}} \underbrace{\mathbb{P}(\mathcal{S}^{F_{t-1}:X_t} | \mathcal{S}^{F_{t-1}:X_{t-1}}, a_{t-1}^{\mathrm{q}})}_{\text{Motion Model}} \boxed{b_{t-1}^{Sp}}$$

$$b_{t-1}^{Sp} \triangleq \underbrace{\mathbb{P}(\mathcal{S}^{\beta_t} | \mathcal{S}^{\mathcal{M}_{t-1}})}_{\text{Composition}} \underbrace{\mathbb{P}(\mathcal{S}^{F_{t-1}} | \mathcal{S}^{\mathcal{M}_{t-1}})}_{\text{Frame Scale}} b_{t-1}$$



Cost Functions

- **Expected number of qualitative states**:

$$c_t(b_t, a_{t-1}) = \mathbb{E}\left[d(\mathcal{S}^{F_{t-1}:X_{t-1}}, \mathcal{S}^{F_{t-1}:X_t}) | \mathcal{H}_t \right]$$

  - Where $d(s_1, s_2)$ represents the minimum number of states traversals required to travel from state s_1 to s_2.

- **Expected Metric Path Length**:

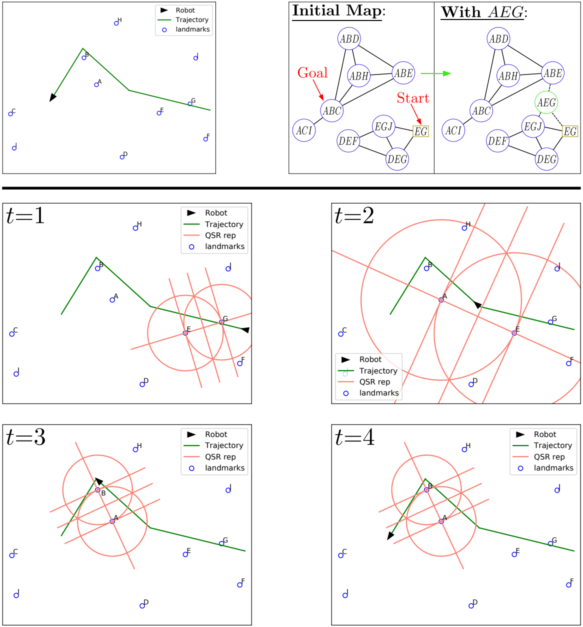
$$c_t(b_t, a_{t-1}) =$$

$$\mathbb{E}\left[\mathbb{E}\left[\left\| \mathcal{X}^{F_{t-1}:X_t} - \mathcal{X}^{F_{t-1}:X_{t-1}} \right\|_2 \cdot \mathcal{X}^{F_{t-1}} | \mathcal{S}^{F_{t-1}:X_t}, \mathcal{S}^{F_{t-1}:X_{t-1}}, \mathcal{S}^{F_{t-1}}, \mathcal{H}_t \right] \right]$$

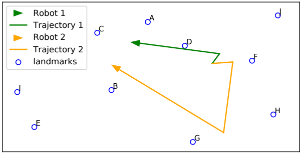
  - Which can be simplified using the Low of Total Expectation:

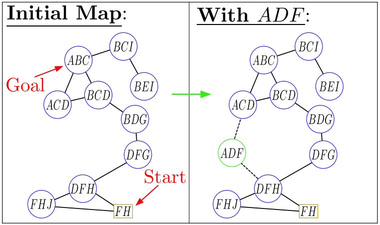
$$c_t(b_t, a_{t-1}) = \mathbb{E}\left[\left\| \mathcal{X}^{F_{t-1}:X_t} - \mathcal{X}^{F_{t-1}:X_{t-1}} \right\|_2 \cdot \mathcal{X}^{F_{t-1}} | \mathcal{H}_t \right]$$

Result example 1



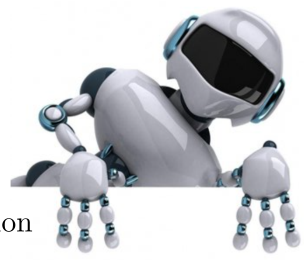
Result example 2 & Some statistics





		Cost 1 (# q-states)		Cost 2 (metric path length)	
		WO Comp	W Comp	WO Comp	W Comp
All Tests (2500)	Plan exists	66%	78.4%	66%	78.4%
	Executed successfully	59.2%	70.8%	60.1%	72%
Comparable & Different Tests (10%)	Average executed cost	6.65	5.33	2.66	2.32

Agenda

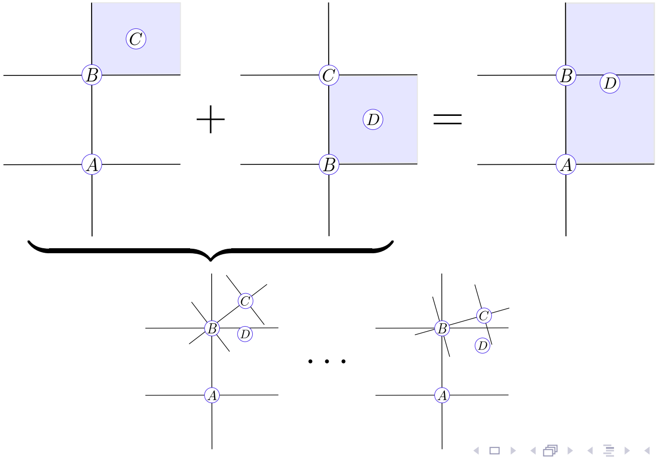


- Introduction and Motivation
- Qualitative Belief Space Planning via Compositions
- Compositions Calculi
- Summary and Conclusions

Reminder - The Composition Operator

- Given two triplets, we can evaluate the third
- Source triplets must share two landmarks in common

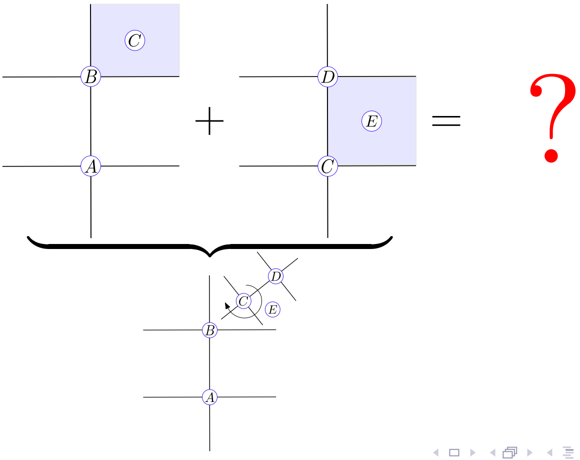
$$\text{Compose}(\text{AB:C, BC:D}) = \text{AB:D}$$



Reminder - The Composition Operator

- Given two triplets, we can evaluate the third
- Source triplets must share two landmarks in common

$$\text{Compose}(\text{AB:C},\text{CD:E}) = {\color{red}?}$$



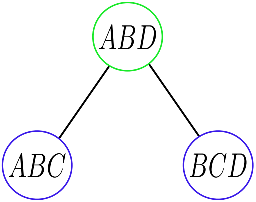
Composition topological Regime - Lemma

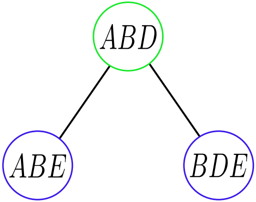
- The following Lemma formulates the above:

Lemma A triplet τ can be composed using a single composition operation (or directly) based on the triplets τ_1 and τ_2, if the following hold:

1) $|\tau_1 \cap \tau_2| = 2$
2) $\tau \subset \tau_1 \cup \tau_2$

- Examples using Composition-Trees representations:

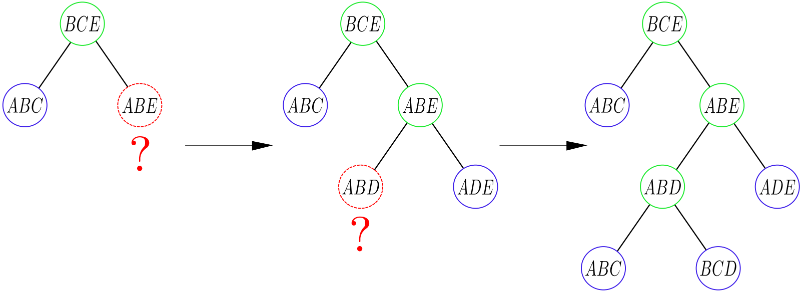




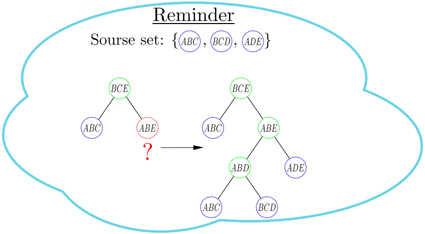
The recursive issue

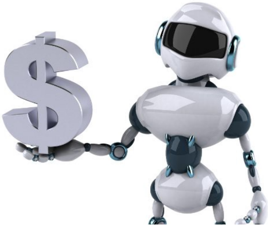
- What if a source triplets required to compose a target one is not available?

Sourse set: $\{ABC, BCD, ADE\}$



Contributions





- We address two questions arising from the above. Given an initial set of source triplets:
  - Q1: What new triplets can be composed?
  - Q2: What is the optimal sequence of compositions operations to create a target triplet?

A paper regarding this part was accepted to RA-L - "Incorporating Compositions in Qualitative Approaches"

Q1: What triplets can be composed?

- We aim to define a sufficient condition on a source set, such that any triplet within the underlying landmark space can be composed.

- **Landmark Space**:

Definition 1. Let $\mathcal{T}$ be a set of triplets. The *Landmark Space* of $\mathcal{T}$, denoted by $\mathcal{L}(\mathcal{T})$, is defined as:

$$\mathcal{L}(\mathcal{T})=\bigcup_{\tau\in\mathcal{T}}\tau$$

$$\mathcal{T}: ABC \quad BCD \quad ADE \longrightarrow \mathcal{L}(\mathcal{T})=\{A,B,C,D,E\}$$

Q1: What triplets can be composed?

- **Cut**:

> **Definition 2.** Let $\mathcal{T}$ be a set of triplets. A *Cut* $C{=}(\mathcal{T}_L,\mathcal{T}_R)$ of $\mathcal{T}$, is a partition of $\mathcal{T}$ into two disjoint subsets, $\mathcal{T}_L$ and $\mathcal{T}_R$, s.t. $\forall\tau{\in}\mathcal{T}$, either $\tau{\in}\mathcal{T}_L$ or $\tau{\in}\mathcal{T}_R$, but not both.

$$\mathcal{T}_L : ABC \quad BCD \qquad \Big| \qquad \mathcal{T}_R : ADE$$

- **α-common Cut**:

> **Definition 3.** Let $\mathcal{T}$ be a set of triplets and let $\alpha{\in}\mathbb{N}{\cup}\{0\}$. A *Cut* $C{=}(\mathcal{T}_L,\mathcal{T}_R)$ of $\mathcal{T}$ is called *α-common* if $|\mathcal{L}(\mathcal{T}_L){\cap}\mathcal{L}(\mathcal{T}_R)|{\geq}\alpha$.

In the example above:

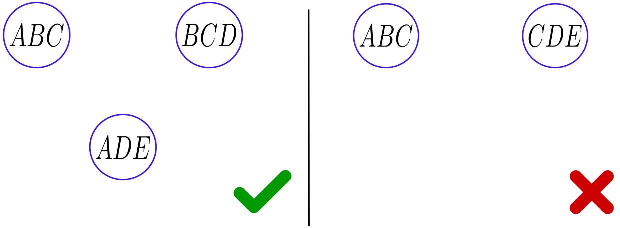
$$|\{\mathcal{L}(\mathcal{T}_L) \cap \mathcal{L}(\mathcal{T}_R)\}|{=}|\{A,D\}|{=}2$$

Q1: What triplets can be composed?

- **Composable set**:

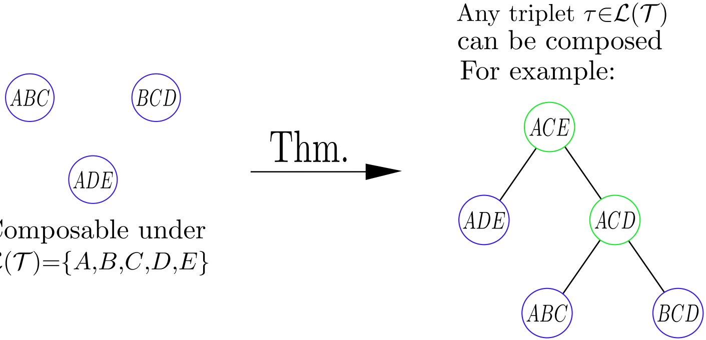
Definition 4. Let $\mathcal{T}$ be a set of triplets and let $\mathcal{L}$ be a *Landmark Space*. We say that $\mathcal{T}$ is *Composable* under $\mathcal{L}$, if $\mathcal{L} \subseteq \mathcal{L}(\mathcal{T})$, and one of the following holds:

1) $|\mathcal{T}|=1$.
2) $|\mathcal{T}|>1$ and **there is** a 2-*common Cut* $C=(\mathcal{T}_L, \mathcal{T}_R)$ of $\mathcal{T}$, s.t. $\mathcal{T}_L$ is *Composable* under $\mathcal{L}(\mathcal{T}_L)$ and $\mathcal{T}_R$ is *Composable* under $\mathcal{L}(\mathcal{T}_R)$.



Q1: What triplets can be composed?

Theorem 1. Let $\mathcal{T}$ be a *Composable* set of triplets under the *Landmark Space* $\mathcal{L}$. Then any triplet $\tau \subseteq \mathcal{L}$ can be composed based on triplets from $\mathcal{T}$.



Q1: What triplets can be composed?

Proof. We prove Theorem 1 using induction on number of set elements (triplets), $|\mathcal{T}|$.
Base step: Suppose $|\mathcal{T}|=2$ ($|\mathcal{T}|=1$ is a trivial case). $\mathcal{T}$ is *Composable* under $\mathcal{L}$, thus, the only non-trivial *Cut* exists in this case is 2-*common*. Without loss of generality (WLOG), suppose $\mathcal{T}=\{ABC,BCD\}$. Indeed, according to Lemma 1, we can compose ABD and ACD, i.e., all other triplets exist in $\mathcal{L}(\mathcal{T})$ (and thus also in $\mathcal{L}$, since $\mathcal{L}\subseteq\mathcal{L}(\mathcal{T})$).
Induction step: Suppose any triplet $\tau\subseteq\mathcal{L}$ can be composed based on triples from $\mathcal{T}$, for all $1\leq|\mathcal{T}|\leq n$. We prove that the same is true for $|\mathcal{T}|=n+1$.

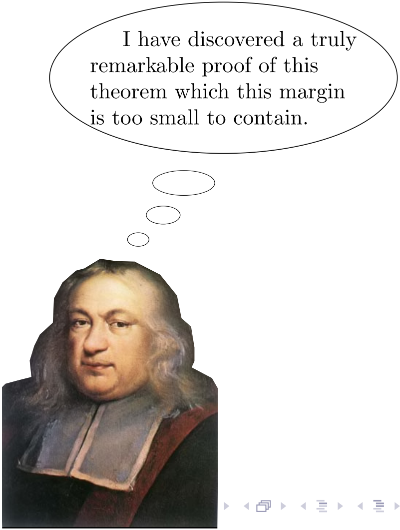
Suppose $|\mathcal{T}|=n+1$ and let $\tau=L_1L_2L_3$ be a triplet in $\mathcal{L}$. We show that τ can be composed using triplets from $\mathcal{T}$. Since $\mathcal{T}$ is *Composable* under $\mathcal{L}$, we are guaranteed that it has a 2-*common Cut*, $(\mathcal{T}_L,\mathcal{T}_R)$, s.t. $\mathcal{T}_L$ is *Composable* under $\mathcal{L}(\mathcal{T}_L)$ and $\mathcal{T}_R$ is *Composable* under $\mathcal{L}(\mathcal{T}_R)$.

Suppose, WLOG, that $\{A,B\}\subseteq\mathcal{L}(\mathcal{T}_L)\cap\mathcal{L}(\mathcal{T}_R)$. We examine three possible cases (see illustration in Fig. 4).

<u>Case 1</u>: $|\{L_1,L_2,L_3\}\cap\{A,B\}|=2$. WLOG, we assume that $L_1=A$ and $L_2=B$ and continue examining L_3. The latter must be in $\mathcal{L}(\mathcal{T}_L)$, or $\mathcal{L}(\mathcal{T}_R)$, or both. WLOG, suppose $L_3\in\mathcal{L}(\mathcal{T}_L)$. Thus, we are guaranteed that $\{A,B,L_3\}\subseteq\mathcal{L}(\mathcal{T}_L)$, and consequently ABL_3 (namely, $L_1L_2L_3$) can be composed according to the assumption since $\mathcal{T}_L$ is *Composable* under $\mathcal{L}(\mathcal{T}_L)$ and $|\mathcal{T}_L|\leq n$.

<u>Case 2</u>: $|\{L_1,L_2,L_3\}\cap\{A,B\}|=1$. WLOG, we assume that $L_1=A$ and continue examining L_2,L_3. If they are both in $\mathcal{L}(\mathcal{T}_L)$ or both in $\mathcal{L}(\mathcal{T}_R)$, we finished (similarly to case 1). Otherwise, WLOG, we assume that L_2 is exclusively in $\mathcal{L}(\mathcal{T}_L)$ and L_3 is exclusively in $\mathcal{L}(\mathcal{T}_R)$. According to the assumption, we are guaranteed that ABL_2 and ABL_3 can be composed based on $\mathcal{T}_L$ and $\mathcal{T}_R$, respectively. Finally, using these two triplets, we can compose AL_2L_3 (Lemma 1), namely, $L_1L_2L_3$.

<u>Case 3</u>: $|\{L_1,L_2,L_3\}\cap\{A,B\}|=0$. If $\{L_1,L_2,L_3\}$ are all in $\mathcal{L}(\mathcal{T}_L)$ or all in $\mathcal{L}(\mathcal{T}_R)$, we finished (similarly to case 1). Otherwise, WLOG, we assume that L_1 and L_2 are exclusively in $\mathcal{L}(\mathcal{T}_L)$ and L_3 is exclusively in $\mathcal{L}(\mathcal{T}_R)$. According to the assumption, we are guaranteed that ABL_2 and AL_1L_2 can be composed based on $\mathcal{T}_L$, and that ABL_3 can be composed based on $\mathcal{T}_R$. Using ABL_2 and ABL_3, we can compose AL_2L_3 (Lemma 1). Finally, using AL_1L_2 and AL_2L_3, we can compose $L_1L_2L_3$ (Lemma 1)$\blacksquare$.



Q2: What is the optimal sequence of compositions operations to create a target triplet?

- We suggest a simple algorithm to address the following problem:

$$T^* = \underset{T \in \mathbb{T}_{\tau_o}}{\arg\min} \sum_{\tau \in T} C(\tau)$$

- The cost function takes the following form:

$$C(\tau) = \begin{cases} C^{\text{source}}(\tau), & \text{if } \tau \text{ is a source triplet} \\ C^{\text{comp}}(\tau_L, \tau_R), & \text{if } \tau \text{ is composed directly using } \tau_L, \tau_R \end{cases}$$

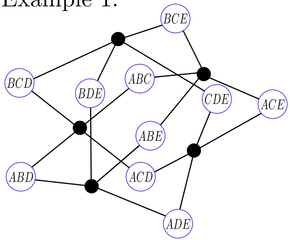
- For example, the unit cost accumulates the number of composition operations required to form a target triplet:

$$C(\tau) = \begin{cases} 0, & \text{if } \tau \text{ is a source triplet} \\ 1, & \text{if } \tau \text{ is composed directly using } \tau_L, \tau_R \end{cases}$$

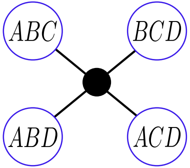
Q2: What is the optimal sequence of compositions operations to create a target triplet?

- **Composition-Graph**:

Example 1:



Example 2:



- The Composition-Graph reflects a direct composition relationship according to the Lemma

Reminder:

> **Lemma** A triplet τ can be composed using a single composition operation (or directly) based on the triplets τ_1 and τ_2, if the following hold:
> 1) $|\tau_1 \cap \tau_2| = 2$
> 2) $\tau \subset \tau_1 \cup \tau_2$

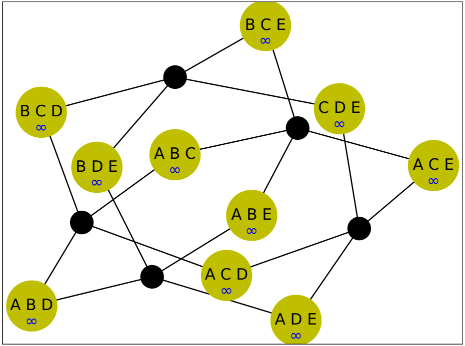
Q2: What is the optimal sequence of compositions operations to create a target triplet?

- Running example:

Landmark Space: $\{A,B,C,D,E\}$

Source set: ABC BCD ADE

Step 1: initialization



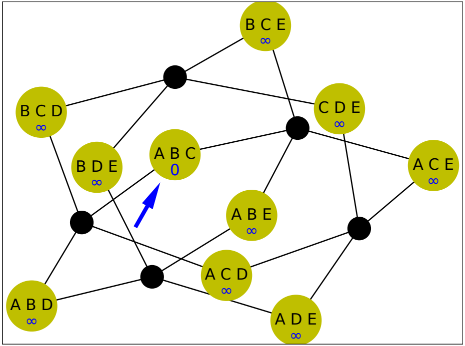
Q2: What is the optimal sequence of compositions operations to create a target triplet?

- Running example:

Landmark Space: $\{A,B,C,D,E\}$

Source set: ABC BCD ADE

Step 2: insert ABC



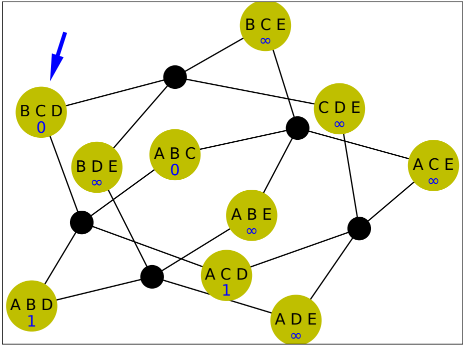
Q2: What is the optimal sequence of compositions operations to create a target triplet?

- Running example:

Landmark Space: $\{A,B,C,D,E\}$

Source set: ABC BCD ADE

Step 3: insert BCD



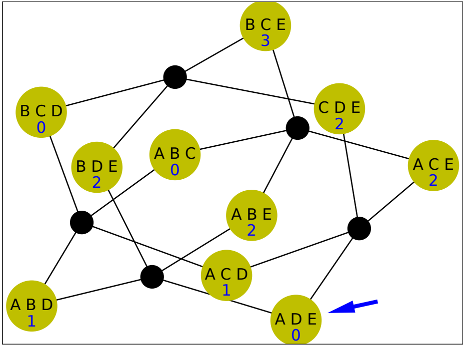
Q2: What is the optimal sequence of compositions operations to create a target triplet?

- Running example:

Landmark Space: $\{A, B, C, D, E\}$

Source set: ABC BCD ADE

Step 4: insert ADE

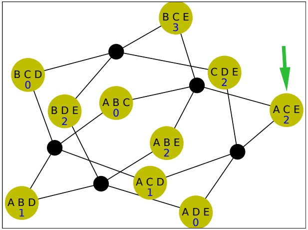


- Running example:

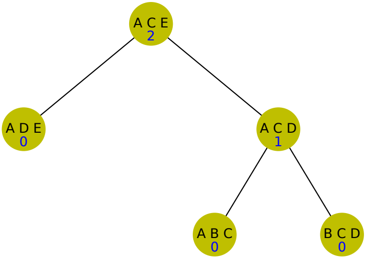
Landmark Space: $\{A, B, C, D, E\}$

Source set: ABC BCD ADE

Step 5: extract ACE composition-tree

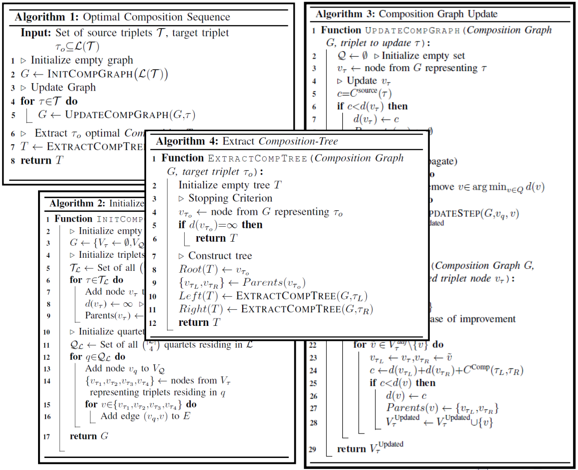


Extract tree

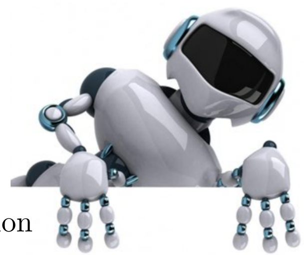


Q2: What is the optimal sequence of compositions operations to create a target triplet?

- For the full algorithm, correctness and complexity analysis, see our paper.



Agenda



- Introduction and Motivation
- Qualitative Belief Space Planning via Compositions
- Compositions Calculi
- $\boxed{\text{Summary and Conclusions}}$

- **Qualitative Belief Space Planning**:
  - A novel Qualitative BSP formulation
  - Compositions incorporation within our algorithm
  - A novel cost function
- **Compositions Calculi**:
  - Composability - a sufficient condition to compose triplets
  - A first-of-its-kind algorithm to find the optimal composition-tree of a terget triplet

Any Questions?